DOCUMENT RESUME

ED 297 941 SE 049 231

AUTHOR	Wiebe, James H.
TITLE	Teaching Elementary Mathematics in a Technological Age.
REPORT NO	ISBN-0-89787-520-6
PUB DATE	88
NOTE	453p.; Some figures and drawings may not reproduce well.
AVAILABLE FROM	Gorsuch Scarisbrick Publishers, 8233 Via Paseo del Norte, Suite F-400, Scottsdale, AZ 85258 ($25.00).
PUB TYPE	Books (010) -- Guides - Classroom Use - Guides (For Teachers) (052) -- Guides - Non-Classroom Use (055)
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Basic Skills; Calculators; Computation; Computer Oriented Programs; Elementary Education; *Elementary School Mathematics; Mathematics Education; *Mathematics Instruction; Number Concepts; *Problem Solving; *Teacher Education; *Teaching Methods

ABSTRACT

This book is based on the underlying philosophy that mathematics learning at the elementary level should be active. Children should learn through meaningful hands-on activities with manipulative materials, pictures and technology. More than 400 pupil activities and suggestions for designing more activities are described. Because mathematics is to be used in real world problem solving, mental arithmetic, estimation and the ability to use technology, the technological activities presented include the use of: (1) calculators for developing understanding of mathematics; (2) programming activities for developing mathematical understandings; (3) application programs such as spread sheets and data base programs for problem solving; (4) calculators and computers in problem solving activities; and (5) tape recorders, overhead projectors, and other devices in the classroom. Standard topics such as whole number computation, fractions, measurement and geometry are also considered. (PK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

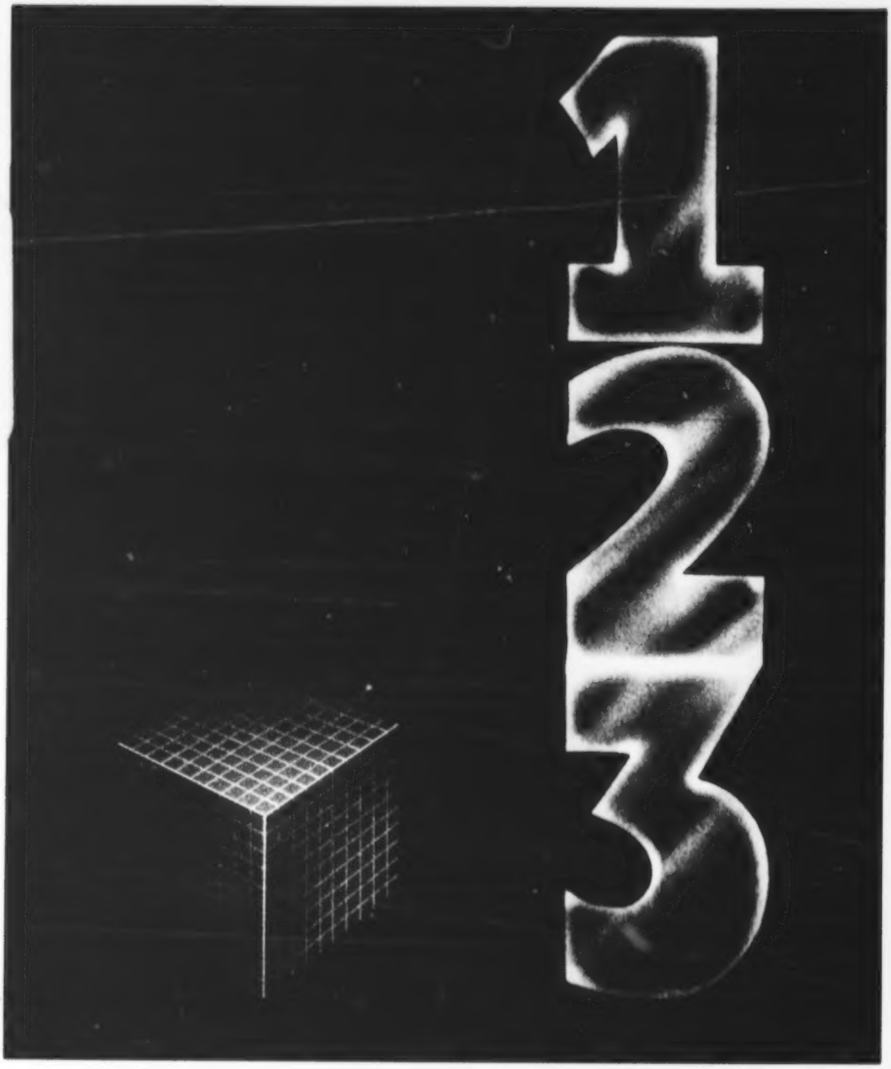
ED297941

TEACHING ELEMENTARY MATHEMATICS
IN A TECHNOLOGICAL AGE

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
☑ This document has been reproduced as received from the person or organization originating it
☐ Minor changes have been made to improve reproduction quality
• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY
John W. Gorsuch

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."



SE 049 231

JAMES H. WIEBE

Teaching Elementary Mathematics In a Technological Age

Teaching Elementary Mathematics In a Technological Age

James H. Wiebe
California State University, Los Angeles

Gorsuch Scarisbrick, Publishers
Scottsdale, Arizona

Editor: *John W. Gorsuch*
Production Manager: *Gay L. Orr*
Manuscript Editor: *Kristine Daines*
Cover Design: *Ted Wolter, Colors Illustration*
Interior Design: *Victoria Vandeventer*
Interior Art: *Carlisle Graphics, Maggie's Fine Art & Graphics*
Layout: *Bill Nebel, Graphic Arts Services*
Typesetting: *Carlisle Graphics*
Printing: *BookCrafters*

Gorsuch Scarisbrick, Publishers
8233 Via Paseo del Norte, Suite F-400
Scottsdale, Arizona 85258

10 9 8 7 6 5 4 3 2 1

ISBN 0-89787-520-6

Copyright© 1988 by Gorsuch Scarisbrick, Publishers

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopy, recording or otherwise, without the prior written permission of the Publisher.

Printed in the United States of America.

PREFACE

This book is about teaching elementary school mathematics. It is based on the underlying philosophy that mathematics learning at the elementary level should be *active.* Rather than simply *listening* to teachers talk, *watching* the teacher write symbolic procedures on the board, and doing *pencil and paper practice,* children should learn through *meaningful hands-on activities* with manipulative materials, pictures, and technology. They should also do a variety of meaningful symbolic activities to develop their understanding of abstract mathematical concepts and their ability to solve problems. This book describes more than 400 pupil activities (identified by ■ in the text) and has suggestions to help teachers design many more.

Another aspect of the philosophy of this book is that mathematics is learned for practical reasons—it is learned to be used in the real world. Thus, problem solving, mental arithmetic, estimation, and the ability to use technology to find answers to mathematical situations are stressed throughout.

This book is designed to prepare students to teach elementary mathematics in the modern world—a world in which calculators, computers, and mental computation are replacing pencil and paper computations on the job and in the home. This book considers that reality when discussing the kinds of mathematics we should teach our children. It also considers how technological tools can be used to teach various concepts and skills in mathematics. A variety of activities involving technology are included so that prospective teachers using the book will gain the experience they need to use technology later in their classrooms. A person taking this course and completing all the computer related activities contained therein will not only have a thorough understanding of how to teach mathematics in the elementary grades, but will also be computer literate. Note, however, that instructors may choose to skip many or most of the computer-related activities in the text without losing the impact of the book's discussions about teaching mathematics in the modern world.

The technological activities presented in this book include the use of the calculator for developing understanding of mathematics, the use of programming activities (in Logo or BASIC) for developing mathematical understandings, the use of applications programs such as spreadsheets and data base programs for problem solving, the use of calculators and computers in problem solving activities, and the use of tape recorders, the overhead projector, and other devices in the classroom.

Some of the unique features of the book include an introductory chapter on Logo and a large variety of suggested Logo activities throughout the text. It also contains a tutorial on introductory BASIC (in an appendix), allowing the instructor to use either BASIC or Logo in programming activities. Or, if the instructor decides to skip the programming activities altogether, there are a variety of other computer-related activities in the text, including four exemplary CAI (computer assisted instruction) programs designed to illustrate the use of CAI in the mathematics classroom. Teachers will be able to use these programs directly in their own classrooms. All four are contained on a disk (one side contains the Apple versions of these programs and the other contains the PC-DOS versions*) that is available from the publisher along with the instructor's manual. Instructors (and students taking the course) are allowed to make as many copies of these programs as desired.

Teaching Elementary Mathematics in a Technological Age offers separate chapters on problem solving, estimation and mental arithmetic, calculators in the classroom, and computers in the classroom. As mentioned above, these topics are also integrated throughout the book. Chapters on classroom management and teacher-designed game/practice activities are also included. Finally, the book contains appendices that provide brief tutorials to AppleWorks spreadsheet and data base programs.

I would like to thank the following individuals for their contributions to this text: Drs. Arthur Wiebe, John Knaupp, Jon Engelhardt, and Gary Bitter for getting me started in the field of Mathematics Education; Jewell Garner for writing several activities for the book; Dorothy Keane, Jon Engelhardt, H. Clifford Clark, and the others for their reviews of the manuscript and suggestions for its improvement; the students, secretaries, and typists at The University of Southwestern Louisiana and California State University Los Angeles for typing and editing the manuscript. Most of all I would like to thank my family—Malorie, Nathan, and Meara—who supported me through the long evenings and weekends while I was working on this manuscript.

---

*Also contained on the disk is a data bank of test items that the instructor can access and manipulate using AppleWorks (for Apple computers) or Lotus (for IBM compatible computers).

CONTENTS

1 **Mathematics in the Modern World** 1

Mathematics vs. Arithmetic 4

Mathematics for the Future 6

Suggested Readings 9

Questions 9

2 **Teaching & Learning Mathematics** 11

Learning Mathematics 12

Teaching Mathematics 21

Suggested Readings 28

Questions 28

3 **The Calculator in the Mathematics Classroom** 31

Teaching Mathematics with Calculators 32

Mathematics Curriculum and the Calculator 40

What Kind of Calculator to Get? 42

Suggested Readings 43

Questions 44

4 **Computers in Mathematics Instruction** 45

Computers and the Mathematics Curriculum 45

Computers and Mathematics Teaching Methods 50

Computer Management of Your Classroom 50

Teaching Mathematics with Computers 55

Computer Hardware 60

Sources of Software 63

Suggested Readings 64

Questions 65

8

5 Teaching Problem Solving in Mathematics 67
Defining "Problem Solving" 67
Teaching Problem Solving 68
Suggested Reading 76
Questions 77

6 Estimation & Mental Arithmetic 79
Setting the Stage 80
Left to Right Computation 81
Front Significant Digits 81
Rounding/Compatible Numbers 83
Fractions and Percentages 86
Averaging 86
Compensation 87
Suggested Readings 88
Questions 88

7 Logo in the Mathematics Classroom 89
Turtle Commands 90
Procedures 94
Switching Between Graphics and Text 97
The Print Command 97
Variables 99
Suggested Readings 102

8 Early Concepts 103
Numeration Systems 104
Activities for Teaching Pre- and Beginning Numbers 105
Sample Drill and Practice Activities 111
Suggested Readings 112
Questions 113

9 Multidigit Numerals 115
Numeration 115
Other Systems of Numeration 117
The Base-Ten, Place Value System 119
Other Base Number Systems 120
Activities for Teaching Numeration 123
Sample Drill and Practice Activities 137
Suggested Readings 138
Questions 139

10 **Addition & Subtraction of Whole Numbers** 141
Addition of Whole Numbers 141
Teaching Addition 142
Properties of Addition 144
Teaching the Addition Properties 146
Subtraction of Whole Numbers 156
Teaching Subtraction 158
Subtraction Principles 163
Sample Drill and Practice Activities 174
Suggested Readings 177
Questions 177

11 **Multiplication & Division of Whole Numbers** 179
Multiplication of Whole Numbers 179
Teaching Multiplication Facts 181
Properties of Multiplication 186
Teaching Multidigit Multiplication 188
Division of Whole Numbers 197
Teaching Division Facts 198
Properties of Division 201
Multidigit Division 202
Sample Drill and Practice Activities 212
Suggested Readings 216
Questions 217

12 **Understanding Fractions & Decimals** 219
Teaching Fractions 221
Teaching Decimals 228
Sample Drill and Practice Activities 236
Suggested Readings 237
Questions 238

13 **Operations on Fractions** 239
Teaching Operations on Fractions 239
Sample Drill and Practice Activities 257
Suggested Readings 258
Questions 260

14 Decimal Operations, Ratios, Proportions, & Percentages 261
Properties of Operations on Rational Numbers 262
Activities for Decimals 263
Calculations with Percentages 277
Sample Drill and Practice Activities 283
Suggested Readings 284
Questions 284

15 Measurement 285
Measurement Systems 285
Teaching Measurement to Children 288
Activities for Teaching Measurement 290
Suggested Readings 304
Questions 304

16 Geometry 307
Geometry in the Curriculum 307
Teaching Geometry 309
Suggested Readings 327
Questions 328

17 Integers, Real Numbers, & Other Topics 331
Teaching about Numbers 331
Suggested Readings 343
Questions 344

18 Data Gathering, Statistics, Probability 345
Teaching Statistics, Graphics, and Probability 345
Suggested Readings 356
Questions 356

19 Assessment, Planning, & Management 357
Assessment 357
Planning 363
Classroom Management 370
Suggested Readings 376
Questions 376

20 Practice Activities & Games in Mathematics 377
Individual Study and Timed Quizzes 377
Group Study of "Facts" 378
Games 378
Designing Your Own Games 380
Computerized Drill 389
Practicing Mathematical Principles 392
Resources for Mathematics Practice Materials 392
Questions and Activities 393

Appendix A An Introduction to BASIC 395
Appendix B AppleWorks Spreadsheets 412
Appendix C AppleWorks Data Base 417
Appendix D The Math Practice Program 423
Appendix E Subtractive Algorithm Program 435
Appendix F Estimating Fractions Program 439
References 445
Index 447

CHAPTER

1

Mathematics in the Modern World

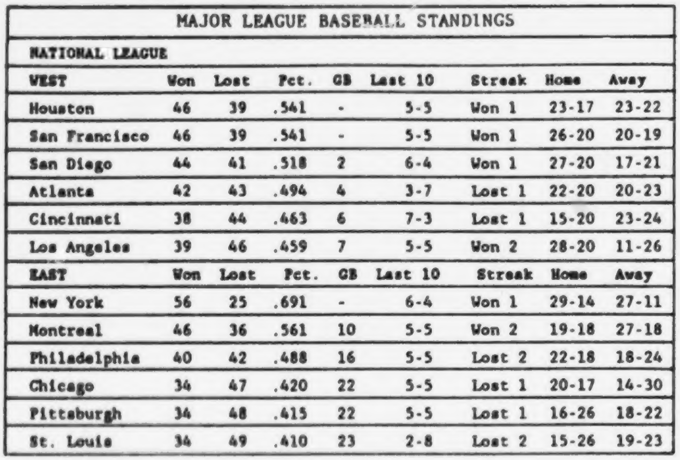
Mathematics is one of the most important tools a person can possess in modern society. It helps us understand and describe the world around us—the food we eat, the clothes we wear, and the cars we drive. When we go to the movies, participate in sports, appreciate art, cultivate our gardens, manage our finances, participate in hobbies, and even when we read novels and newspapers, we use mathematics.

Mathematics has been important as long as humans have lived in civilized societies. In the beginning, it was used mostly in keeping track of possessions, in bartering and trading, in building and architecture, and in navigation. As our society became more urbanized and industrialized, math came to be an integral part of everyday existence. Today, engineers, computer scientists, actuaries, builders, business people doing market surveys, and many others use high-level mathematics every day. Clerks, janitors, baseball players, salespeople, disk jockeys, and people in almost any job you can name, use math (mostly arithmetic) frequently. For a typical person living in North America or Europe, the mathematics of bank accounts, income taxes, bills, sales in department stores, automobiles, houses, and gardens is inescapable.

The technological devices we use every day, such as radios and microcomputers, abound with mathematical applications. In order to find our favorite radio station on the dial, we need to be able to read and understand base ten numerals. If we wish to write computer programs even at a beginning level, we need to understand basic arithmetic operations, variables, exponential notation, and coordinate graphing. For computer programming at a higher level, we need to know trigonometry, logarithms, functions, and many algebraic concepts. Understanding how a computer works and programming in machine language requires an understanding of base two, base eight, and perhaps base sixteen number systems. With computers becoming so all-pervasive in our world, some knowledge of computers and the related mathematics is becoming necessary for success, advancement, and perhaps even survival.

Even something as simple and nontechnological as an orange involves mathematics. A person who grows oranges is concerned about the ratio of orange juice to waste (peel, seeds), the sugar content of the juice, the shape of the orange, and

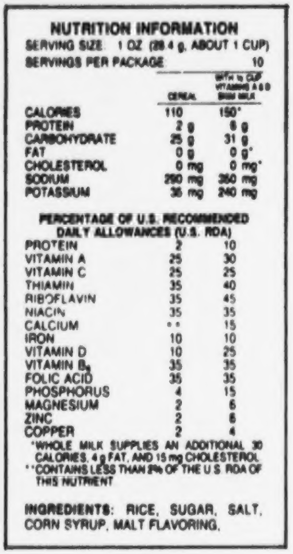
FIGURE 1.1
Using mathematics in everyday life.



MAJOR LEAGUE BASEBALL STANDINGS

NATIONAL LEAGUE

WEST	Won	Lost	Pct.	GB	Last 10	Streak	Home	Away
Houston	46	39	.541	-	5-5	Won 1	23-17	23-22
San Francisco	46	39	.541	-	5-5	Won 1	26-20	20-19
San Diego	44	41	.518	2	6-4	Won 1	27-20	17-21
Atlanta	42	43	.494	4	3-7	Lost 1	22-20	20-23
Cincinnati	38	44	.463	6	7-3	Lost 1	15-20	23-24
Los Angeles	39	46	.459	7	5-5	Won 2	28-20	11-26
EAST	Won	Lost	Pct.	GB	Last 10	Streak	Home	Away
New York	56	25	.691	-	6-4	Won 1	29-14	27-11
Montreal	46	36	.561	10	5-5	Won 2	19-18	27-18
Philadelphia	40	42	.488	16	5-5	Lost 2	22-18	18-24
Chicago	34	47	.420	22	5-5	Lost 1	20-17	14-30
Pittsburgh	34	48	.415	22	5-5	Lost 1	16-26	18-22
St. Louis	34	49	.410	23	2-8	Lost 2	15-26	19-23



NUTRITION INFORMATION
SERVING SIZE: 1 OZ (28.4 g, ABOUT 1 CUP)
SERVINGS PER PACKAGE 10

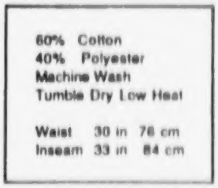
	CEREAL	WITH ½ CUP VITAMINS A & D SKIM MILK
CALORIES	110	150*
PROTEIN	2 g	6 g
CARBOHYDRATE	25 g	31 g
FAT	0 g	0 g*
CHOLESTEROL	0 mg	0 mg*
SODIUM	290 mg	350 mg
POTASSIUM	35 mg	240 mg

PERCENTAGE OF U.S. RECOMMENDED DAILY ALLOWANCES (U.S. RDA)

PROTEIN	2	10
VITAMIN A	25	30
VITAMIN C	25	25
THIAMIN	35	40
RIBOFLAVIN	35	45
NIACIN	35	35
CALCIUM	**	15
IRON	10	10
VITAMIN D	10	25
VITAMIN B_6	35	35
FOLIC ACID	35	35
PHOSPHORUS	4	15
MAGNESIUM	2	6
ZINC	2	6
COPPER	2	4

*WHOLE MILK SUPPLIES AN ADDITIONAL 30 CALORIES, 4 g FAT, AND 15 mg CHOLESTEROL
**CONTAINS LESS THAN 2% OF THE U.S. RDA OF THIS NUTRIENT

INGREDIENTS: RICE, SUGAR, SALT, CORN SYRUP, MALT FLAVORING.



60% Cotton
40% Polyester
Machine Wash
Tumble Dry Low Heat

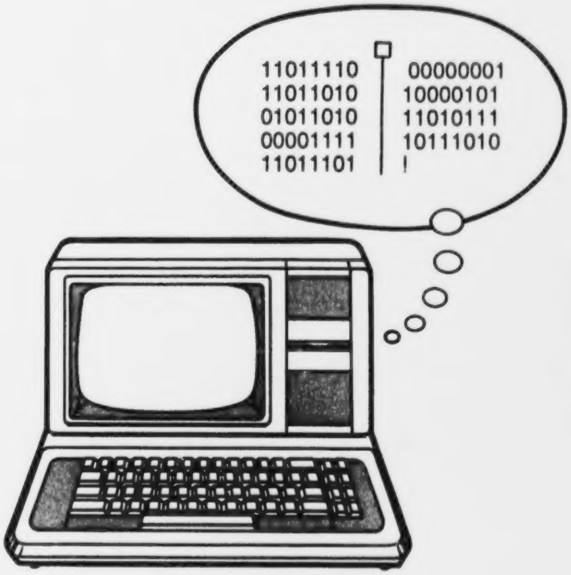
Waist 30 in 76 cm
Inseam 33 in 84 cm

so on. An artist might be interested in the symmetry of the orange, its color, and its shape. If you sat and thought about it for awhile, you could probably list 50 or so types of mathematics related to this simple object. Aspects such as surface area, number of seeds, number of wedges, volume, and weight of the peel would be only a beginning.

The reason for talking about microcomputers and oranges at the beginning of an elementary mathematics methods textbook is so that you will rethink your views about what mathematics is and how it should be taught. Sometimes we become so involved in the day-to-day routine of our jobs that we forget the reasons why we teach some aspects of the subjects in our programs. Prospective and practicing teachers need to examine their preconceived notions about learning, many of which were formulated during their own elementary, secondary, and college general education. We may find ourselves teaching certain mathematics topics because "they are a required part of the curriculum" or because "they are in the textbook." Our justification for the topics we teach and the methods we use to teach them should be stronger than that. We need to be able to explain our practices to ourselves, to parents, to administrators, and not least of all to our students.

Many people believe that elementary mathematics consists of the pencil and paper computations that we all learned in school, such as "long" multiplication or inverting and multiplying when dividing one fraction by another. As will be discussed in Chapter 2, these are only two of a large number of possible algorithms (series of steps that will give you a correct answer to a mathematics "problem") for doing the same operations. These two algorithms have served our society well for a

FIGURE 1.2



hundred or so years. However, with the explosion of the use of calculators and computers, the ability to do pencil and paper algorithms is a dying art. With calculators in most everyone's briefcase, pocket, or purse, why should we spend a year or more teaching the mechanics of long division? There are other ways of dividing that are much easier to remember and can be taught in a fraction of the time we spend teaching long division.

We need to examine what we are teaching to children to make sure that it is appropriate for the 1990s, the 2000s, the 2010s—the years in which our students will be using the skills they have learned in their jobs and everyday lives. For example, it is important that we teach children to become good *estimators*. Using this skill, they can determine if the numbers they read on their calculators or cash registers are accurate. Of course, pencil and paper computations will not be completely abandoned—they will still be used for teaching, for on-the-spot calculations with numbers too large for mental computations, when the answer must be exact and no electronic device is near and so on.

The adjustments we make because of the new technology will be mostly in the arithmetic we teach children. As mentioned, we will place less emphasis on long, tedious computations using pencil and paper and more emphasis on estimation, meaning, mental computation, short pencil and paper algorithms, and problem solving. Mathematics instruction will change very little as a result of new technology, however. Mathematics is the logical study of shapes, patterns, arrangements, quantities, and other ideas associated with these. The basic fundamentals do not change much from year to year, although there is some change in the topics we choose to emphasize and include in our curriculum as adult job skills change. In 2012, our students will, however, have as much need to understand the meaning of written numerals, fractions, geometric shapes, and basic principles such as the commutative property or the identity property as they did in 1980. In fact, with our society becoming increasingly technological and with more and more jobs based in technical areas, the importance of mathematics will grow while the need

to be able to do long, speedy computations will diminish. This textbook is about teaching mathematics for the modern, technological world. It is future-looking—designed to prepare you, the prospective or practicing teacher, to teach mathematics to students who will begin full-time employment between 5 and 20 years from now, rather than today or 20 years ago.

MATHEMATICS VS. ARITHMETIC

A book on mathematics methods has two primary concerns: curriculum and instruction. Curriculum is the *what:* Should we teach geometry in the second grade? Should we teach bases other than base ten? Before the 1960s, elementary math curriculum focused on the four basic operations (addition, subtraction, multiplication, and division), on whole numbers, fractions, and decimals, and on word problems involving these operations. In other words, elementary school children learned arithmetic. Geometry, algebra, mathematical principles, bases, and other topics were saved for later grades.

After Russia launched the first artificial satellite, Sputnik, there was a great deal of concern about upgrading mathematics and science instruction in the United States. At that time, mathematicians and mathematics educators began developing a stronger curriculum designed for the 20th century. Organi[illegible]round the structure of mathematics, its interconnections and fundamental principles, this new curriculum emphasized the solving of problems useful to the learner and society, and it conveyed an appealing and intellectually stimulating view of mathematics. Advanced mathematical concepts, many of which studen[illegible] previously studied only in college, if at all, were now taught in simplified ways that children could understand. New curriculum content included sets, systems of numeration, properties of operations, bases other than base ten, geometric concepts, and organizing and interpreting data. Perhaps most importantly, there was an increased emphasis on helping students develop an *understanding* of mathematics, rather than just computational skills.

Beginning in the mid-1970s, however, there was a move to get "back to the basics." This move was initiated not by mathematics educators but by the general public. It was based on several factors, including what was perceived as the "failure" of the new math, declining standardized test scores, the social unrest of the late 1960s and early 1970s, and a conservative trend in society. Many people blamed falling test scores on the curricula and methodology used during the new math period. In reality, the declining scores were probably due more to factors such as society's changing expectations of what schools should accomplish (e.g., racial integration, socialization, day care, and so on), and the disruptions caused by the implementation of these programs, by legislative and court intervention in our schools, and by changing family structures.

The result of the back-to-basics movement was a series of laws passed in state legislatures designed to ensure that students knew their "basic facts" and could do rapid and long pencil and paper computations before receiving a junior high or high school diploma. Textbooks abandoned sections on functions, set theory, and base five arithmetic and began to include more computational exercises.

At the same time, certain researchers speculated that one reason why children had a difficult time with word problems was that they could not read the problems. It is the author's opinion that as a result of this speculation, textbook publishers and authors removed most of the written text from elementary mathematics textbooks. If you look at a typical pre-1960 mathematics textbook, you will see many pages of problem-solving activities and written explanations of mathematical principles. Some recent elementary mathematics textbooks are almost devoid of reading materials. It is interesting to note that researchers have since found that difficulties with word problems relate not so much to reading problems as they do to lack of understanding of the mathematical concepts involved.

Furthermore, children need frequent and continued exposure to text that contains mathematical content so that they will be able to handle the type of mathematics they will encounter in later life. If such experiences are not provided in textbooks, teachers should frequently require students to read and do activities with newspaper articles and other appropriate text containing mathematical ideas.

The latest trends in mathematics are an attempt to tie mathematics to real-life applications and problem solving, to teach for understanding, and to integrate calculators and computers into the math curriculum. The new mathematics has not been abandoned by any means. Much of the curriculum in modern textbooks was first introduced during the new math epoch. Topics still present from the 1960s are (1) the use of sets to introduce number concepts, (2) mathematical principles such as the commutative property of addition, (3) geometric concepts, (4) measurement, and (5) methods for solving nonroutine problems. Textbook authors still attempt to unify mathematics, to investigate mathematical patterns, and to prepare students for more advanced mathematical topics. Gone, however, are the more ostentatious trappings of the period—subject matter that was hard to justify in relationship to what children would need for real-life applications and the later study of mathematics. Number systems other than base ten, formal set language, and the study of logic were the first to go. Note, however, that because of the computer, other numeration systems (especially bases 2, 8, and 16) are making their way back into some textbooks.

MATHEMATICS FOR THE FUTURE

What mathematics will your students need when they begin work, 5 to 17 years from now? We have already stated that the world of the present and near future places little value on the ability to do complex pencil and paper computations but requires instead the ability to do mental arithmetic, to use electronic devices for more complex computations, and to solve problems. Many professional teacher organizations, state departments of education, and other bodies influencing curriculum and methods in elementary classrooms have recognized the need to organize curriculum and instruction with an orientation toward future applications rather than toward the past.

In 1985, the National Council of Teachers of Mathematics, in an article in the *Arithmetic Teacher,* detailed the recommendations from a conference funded by the federal government on the impact of calculators and computers on school mathematics (Corbitt 1985). Numerous other articles in the *Arithmetic Teacher* have centered on this topic (for example, the January 1987 issue on calculators, and their 1984 Yearbook, *Computers in Mathematics Education*). States such as California (*Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve,* 1986) and Texas have revamped curriculum guidelines for mathematics based on the new technologies. The list of changes in scholarship and curriculum is far too great to include here.

Perhaps the most influential of the publications directed at changing the focus of mathematics instruction was published by the National Council of Teachers of Mathematics in 1980. Although it is several years old, *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (1980) is still relevant and continues to have a strong influence on more recent publications, such as California's Department of Education *Mathematics Framework* (1986), that will in turn influence the content of elementary mathematics textbook series. It offers a blueprint for mathematics curricula designed to prepare students for the modern world. The recommendations outlined in that document that are relevant to this text are paraphrased below:

- **Problem solving must be the focus of school mathematics in the 1980s.** Problem-solving activities and instruction should include not only the routine, textbook word problems but also the nonroutine processes that are essential to the day-to-day living of every citizen (discussed in Chapter 5). The emphasis of our mathematics instruction should be on developing the ability to apply mathematical skills and understanding. At the same time, we need to build a solid foundation of mathematical principles, teaching, for example, the structure of our number system and the commutative property upon which problem-solving proficiency can be built.

- **The concept of basic skills in mathematics must encompass more than computational facility.** Many in our society have the mistaken notion that "basic skills" in mathematics involve mostly pencil and paper computational ability. In practice, these skills involve much more. In a society of economic and fiscal complexity, urbanization, population explosion, electronic communications, and data processing, skills once considered essential have been replaced with new skills. It is time-consuming and costly to make all students proficient in such paper and pencil computations as "3841 × 937" or "72,509 ÷ 29.3". Our time could be better spent on the skills of problem analysis and interpretation. Teaching children to use the calculator for rapid and accurate computation with larger numbers will make a far greater contribution to their functional competence in daily life. We should look for a balance between mental computations, pencil and paper algorithms, and the use of a calculator.

According to the National Council of Supervisors of Mathematics (*Position Paper on Basic Skills*), the ten basic skills in mathematics are as follows:

- Problem solving
- Applying mathematics in everyday situations
- Alertness to the reasonableness of results
- Estimations and approximation
- Appropriate computational skills
- Geometry
- Measurement
- Reading, interpreting, and constructing tables, charts, and graphs
- Using mathematics to predict
- Computer literacy

To this list, we add an eleventh skill:

- Using technological devices to solve mathematical problems.

According to the National Council of Teachers of Mathematics, increased emphasis in our math classrooms should be placed on the following areas:

1. locating and processing quantitative information
2. collecting data
3. interpreting data
4. drawing inferences and predicting from data
5. estimating measures
6. measuring using appropriate tools
7. mentally estimating results of calculations
8. calculating with numbers rounded to one or two digits
9. using technological aids to calculate
10. using ratio and proportion to deal with rate problems in general and with percent problems in particular
11. using imagery, maps, sketches, and diagrams as aids to visualizing and conceptualizing a problem
12. using concrete representations and puzzles that aid in improving the perception of spatial relationships

The NCTM recommends a deemphasis on isolated drill, performing paper and pencil calculations with numbers of more than two digits, mastering highly specialized vocabulary not useful later in mathematics or in daily living, converting measures given in one system to corresponding measures in another, and working with tables whose usefulness has been supplanted by calculators (for example, square root tables).

- **Mathematics programs must take full advantage of the power of calculators and computers (and other media) at all grade levels.** Although explanation, practice, and directive teaching are important, they should not overshadow meaningful problem-solving activities by children. Teachers should use a variety of instructional strategies, materials, and resources to teach mathematics, including individual and small-group work, media (such as overhead projectors, videotapes, video disks, audio/video cassettes, computers, films, slides, and television), discovery and inquiry as well as basic drill, manipulatives to illustrate or develop concepts and skills, review of past topics, and materials and references outside the classroom.

- **The success of mathematics programs and student learning must be evaluated by a wider range of measures than conventional testing.** Evaluation of our students and programs should not be limited to traditional testing. We should include evaluation of such things as children's time on task and their skill in problem-solving strategies and error strategies. If test scores are the sole index of achievement, the public assumes that the sole objective of schooling is a high test score (whether or not what is being tested fits what students should learn).

- **More mathematics study must be required for all students and a flexible curriculum with a greater range of options should be designed to accommodate the diverse needs of the student population.** The amount of mathematics most students take does not compare with the importance of mathematical understanding to their lives. In the elementary grades, *each child* should be given a daily, substantial block of "quality" mathematics instruction at a time of day when learning may be maximized. At the high school level, students planning to go to college should take three or four years of mathematics. Recent surveys have shown that as many as 80% of the "professions" require at least a semester of college level calculus and the chances of a student who does not take four years of high school mathematics taking college calculus are remote.

- **Mathematics teachers must demand of themselves and their colleagues a high level of professionalism.** Practicing teachers should attend mathematics and other professional meetings, read professional journals and learn to use new technologies. Teachers' professionalism demands taking the time and effort to teach mathematics so that students will understand it and be able to apply it in their lives.

These recommendations serve as a springboard for the rest of this book. Throughout the book, problem solving, technology, development of meaning through models and alternative algorithms, and estimation skills are continuing themes. Before talking about methods for teaching skills and concepts in mathematics, however, a larger foundation must be laid. Specifically, you need to know how children learn mathematics and have some general ideas about using calculators and computers in the classroom and teaching for problem solving. Chapters 2, 3, and 4, respectively, address these issues.

SUGGESTED READINGS

An Agenda for Action: Recommendations for School Mathematics of the 1980s. Reston, VA: National Council of Teachers of Mathematics, 1980.

Arithmetic Teacher (Special Calculator Issue) 34 (5), January 1987.

Computers in Mathematics Education (1984 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1984.

Corbitt, M. K. The Impact of Computing Technology on School Mathematics: Report of an NCTM Conference. *Arithmetic Teacher, 32* (8) April 1985: 14-18.

Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve. California State Department of Education, 1986.

Toffler, A. *Learning for Tomorrow: The Role of the Future in Education.* New York: Vintage Books, 1974.

QUESTIONS

1. List 5 mathematical applications for each of the following:
   a. a football game
   b. cooking
   c. an automobile
   d. your hobby
2. Ask several people who have calculators what types of math they do with them. What types of mathematics do they do using pencil and paper? How has their use of math changed since they got a calculator?
3. Suppose that you were teaching multiplication facts (1-digit times 1-digit multiplication) to your class. Describe two activities involving multiplication based on football that you could do with your class.
4. Describe a classroom activity involving multiplication of fractions and involving cooking that you could do with a sixth grade class.
5. Look at the advertisement brochures or introductory documentation (for example, operating instructions or beginning programming manuals) of popular microcomputers like the Apple or IBM personal computers. What mathematics is required to be able to understand these materials?
6. Decode the message in Figure 1.2.

CHAPTER

2

Teaching & Learning Mathematics

When my first child was a few months old, I remember thinking that it was about time he learned to crawl. He certainly had the strength and coordination to crawl; he could lift himself to his hands and knees and push himself around the house endlessly on his stomach. His method of locomotion, however, was very inefficient compared with crawling. Knowing that the next stage of development for him was to crawl, and being a teacher, I endeavored to teach him. My first approach was the didactic method. This involved explaining to him what crawling was, then exhorting him, "Crawl, Nathan," while I melodramatically crawled around the room. When this did not produce the desired results, I tried peer teaching: I brought over a friend's son, Mikie, who was a few months older and a proficient crawler, to demonstrate the desired technique. Unfortunately, Nathan was more interested in trying to hit Mikie with a toy truck than in imitating his marvelous method for locomotion. Finally, I tried the concrete method. My skeptical wife and I moved his arms and legs in the proper crawling motion, guiding him giggling and drooling around the carpet.

My efforts to teach my son to crawl were totally fruitless. He continued creeping merrily around the house until one day, quite by accident, he taught me a new method of learning—the discovery method. About three weeks after my "lessons," as Nathan was in the process of getting up on his knees, lunging forward, and falling on his face, he inadvertently moved his arm forward at the same time he was pushing off from his knees. When he realized that he had moved forward without falling on his face, he tried again and by the end of the day he was crawling unsteadily, and within a few weeks, he was a very proficient crawler.

A few years ago, when I was employed in a clinic helping students who were having problems learning math, I worked with a sixth-grade student who could not seem to understand his teacher's presentations about fractions, decimals, and percentages. He continually did poorly on exams, and could not successfully complete his assignments. After doing extensive diagnostic work, I discovered that he had no understanding of what fractional numerals were. To him, "⅓" had no meaning. The same was true about decimal numerals and percentages. It appeared that his teachers had always explained fractions and decimals verbally and with symbols, and he had never understood what fractions were or how they are manipulated in mathematical operations. After about thirty minutes of instruction using concrete objects and drawn and shaded shapes on paper to represent fractions, the light dawned and he exclaimed, "So that's what those numbers mean!" For the next three months, we met twice a week. I continued to stress meanings, using manipulative materials and pictures, and we studied operations on fractions and decimals. Soon he was working near grade level in mathematics, understanding his teacher's presentations, and obtaining satisfactory marks on his work.

We sometimes assume that children receive and interpret new information in the same way adults do—primarily through language and symbols—and gear our instruction accordingly. This is a major cause for failure to learn mathematics and other subjects. It is vital that we know something about the students whom we are trying to teach. It is also helpful to be conscious of our own preferred teaching style so that we can be assured that the messages we send to students are received and put to use. In this chapter, we will briefly examine the way children learn. Using this foundation, we will investigate ways to ensure the maximum possible learning and performance in mathematics. Since this is a book about teaching mathematics in the elementary (K–8) grades, we shall limit our discussion of learning to those ages.

LEARNING MATHEMATICS

Learning Through Concrete Experiences

There is obviously a large difference between the way newborn infants learn and the way adults learn. Since infants understand neither spoken and written language nor other abstract symbols, they must learn through physical experiences. Most adults, on the other hand, learn primarily through language and symbols. In the lower elementary grades, physical and social activity are extremely important and, as children grow older, verbal and symbolic experiences become increasingly important.

The importance of physical learning cannot be overemphasized. Learning theorists such as Jerome Bruner and Jean Piaget, among others, have emphasized the importance of physical experiences and interactions with the environment in the early learning process. According to Furth (1970), Piaget believed that although there is a close relationship between language and thought at the highest level of development (the level of the propositional or hypothetico-deductive thought stage), this stage is not reached until approximately the age of twelve. At earlier ages, childrens' physical actions with their environment and their social interactions develop their mental structures.

Bruner (1957) believed that man categorizes his environment in order to reduce its complexity and thus to understand it better. Part of the categorization process involves representation; humans represent their environment in a variety of ways. One is through action. For example, a young child represents an automobile by manipulating a block of wood or drawing on a piece of paper. Another way we represent the environment is through imagery. For example, children build and manipulate mental images when they engage in imaginative play. Finally, we use words and language and other symbols (such as algebraic sentences) to help ourselves understand the world.

Bruner suggested that complex concepts in mathematics be introduced to young children in simple forms, at their level. The commutative property, for example, could be taught by representing numbers and the operations on the numbers (for example, addition or multiplication) with objects like colored chips. He suggests that students should be provided with simple apparatus that they can manipulate in order to discover mathematical principles and solutions. Next, children should be encouraged to form perceptual images of the mathematical ideas, and, finally, they should develop a notation for describing the operation (Bruner 1971).

One of the more influential mathematics educators of this century, Zolton P. Dienes, bases his theory of instruction partially on Piaget's findings and theories. He emphasizes the importance of developing meaning for mathematics, the active approach to mathematics instruction, and learning aids. He states that teachers should take the necessary aids from the child's environment and build artificial situations from which the child will be able to abstract the concepts required.

> Structured aids are usually designed and invented by experts who are aware of the concepts which must be embodied in a concrete way. They are so constructed that the child using them will be led towards the abstraction of each desirable concept in turn, with the possibility of controlling the extent of irrelevant qualities in the aids themselves, or in the situation in which they are used....To sum up: with a few of the principles suggested by Piaget, we would include the need for the child to learn by activity, through his own personal experiences, (and) with the use of physical objects, preferably structured aids when the concepts involved are of a higher order so that the situations embodying them are not commonly available in everyday life. (Dienes 1971, 54–55)

Another important aspect of Dienes' approach to teaching mathematics is multiple embodiment—representing a concept in two or more different ways. Because of the differences in learning among children, no one can be sure that a specific structural aid will communicate the fundamentals of a concept to a child. "The more different we make the various embodiments available, the more likely we are able to cater to every child who wishes to learn. Therefore, each child should be exposed to as many of these different embodiments as possible" (Dienes 1971, 56).

An example of the principle of multiple embodiment is the use of two different devices for teaching subtraction in problems like "twenty-five take away nine" to first or second graders. One embodiment of this problem—using objects that appear in the child's environment—would be to represent the minuend with 25 dried beans and to remove nine beans, then to count the result. Another embodiment would be to use a structured aid such as a place value chart. Two tens and five ones are represented in proper columns on the chart (see Figure 2.1). Next, a ten is exchanged for ten units. Next, nine units are removed to show the operation. Finally, the answer (1 ten and 6 units or 16) is noted.

A common thread in the work of all the above-mentioned learning theorists is the importance of developing meaning when teaching mathematics. Mathematics should not be seen as a series of unrelated facts and procedures to be memorized and mentally filed away for later tests. Rather, mathematics is a set of interrelated principles (multiplication, for example, can be thought of as repeated addition) that can be applied in a variety of ways (such as using multiplication or repeated addition to determine how many seats are in the school auditorium). In elementary-

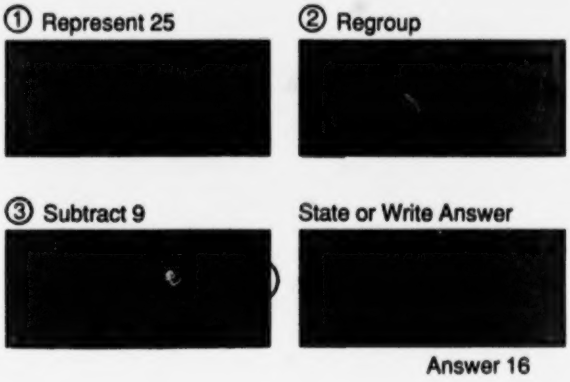


FIGURE 2.1
A model for 25 − 9.

school mathematics, the use of concrete objects by students in learning tasks is directly related to the development of meaning (Weaver and Suydam 1972).

Other mathematics learning theorists have supported the idea that concepts should be presented first at the concrete level. Van Engen (1949) stated that

> In any meaningful situations, there are always three elements. (1) There is an event, an object, or an action. In general terms, there is a *referent.* (2) There is a symbol for the referent. (3) There is an individual to interpret the symbol as somehow referring to the referent. . . . It is important to remember that the symbol refers to something outside itself. This something may be anything whatsoever, even another symbol, subject only to the condition that in the end it leads to a meaningful act or mental image. (323)

In this statement, Van Engen recognizes the basic supposition that there is a learning continuum from concrete to abstract. However, he also implies that concrete materials are not essential to every stage in the process of learning. At some point, symbols may serve as referents for further learning (Suydam and Higgins 1977).

Numerous researchers have investigated the hypothesis that the use of manipulative materials will result in higher achievement than when strictly symbolic approaches (verbal language and written symbols) are used. After reviewing 23 well-designed studies that investigated this hypothesis, Suydam and Higgins concluded that "non-manipulative materials cannot be expected to produce superior achievement in elementary-school mathematics. . . . The use of manipulative materials has a high probability of producing greater mathematics achievement than do non-manipulative sequences" (1977, 18). They say further that there appears to be no evidence that manipulatives are any less beneficial in the upper elementary grades than in the lower grades.

In the above discussion, a variety of terms have been used to describe the materials used to teach mathematics to children—physical objects, concrete materials, structural aids, and manipulative materials. In this book, the term *model* will be used as synonymous with the specific terms above. We define a *model* as

> *a simplified representation of a mathematical concept or operation designed to communicate this idea to the learner.*

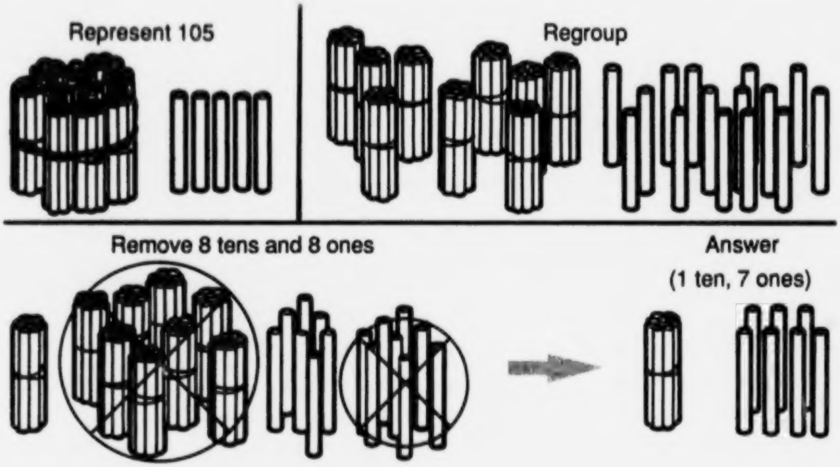


FIGURE 2.2
A bundled stick model for 105 − 88.

Models are used throughout our society. A model airplane built by an elementary pupil is a simplified representation of the real thing. It communicates some aspect of the real object to someone. Models of human genes are designed to communicate certain aspects of unseen natural phenomena to a student or researcher. An architect builds a model of a building to judge its appearance, its structural soundness, and so forth.

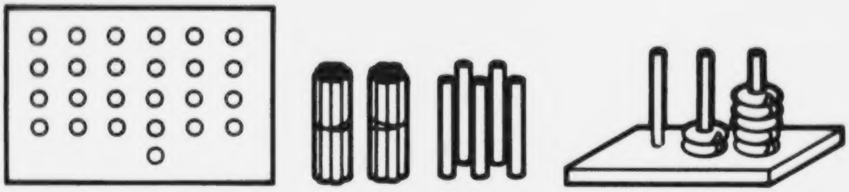
Models may be physical or pictorial. Physical models are objects that can be manipulated to illustrate certain aspects of mathematical ideas; thus, they are sometimes called "manipulative materials." For example, bundled sticks that are made from popsicle or stirring sticks and rubber bands are grouped by tens or ten tens (hundreds) to represent multi-digit numbers and operations on them. Figure 2.2 shows how they can be manipulated to represent the operation, 105 − 88.

Pictorial models are two-dimensional representations of mathematical ideas. In general, pictorial models cannot be manipulated, though the user may draw arrows, circles, or make marks to represent certain operations like addition or regrouping. An example of a pictorial model is dots drawn in an array to represent a product of two single-digit numbers.

When using models to develop meaning for mathematical concepts, it is a mistake to think, "A model is a model is a model." All models are not equal. Most ideas and operations in arithmetic can be represented in a variety of ways. Figure 2.3 shows three ways to represent the number 25. The first model is the least abstract because it represents only the numerousness of the numeral 25. The second model attempts to represent the grouping concepts inherent in the numeral—the two bundles of ten represent the value of the 2 in 25. The third model is the most abstract because it represents the place-value aspects of 25.

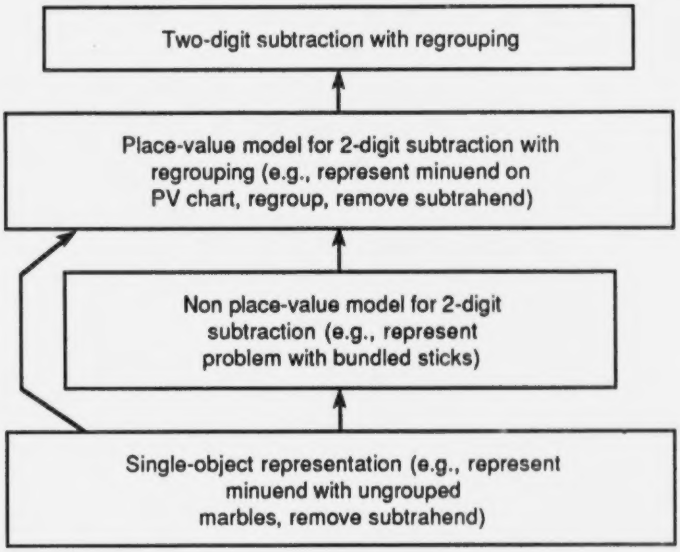
Figure 2.4 shows a hierarchy of models for two-digit subtraction with regouping. Note that the bottom step (representing subtraction with single objects) is essential—all children should experience this level at some time, whether in or out of class. Most children need to experience one of the next two levels (using bundles of sticks or place-value chart). Some children may be able to jump from the lowest level to symbolic subtraction with ease; however, such instances are rare. Figure 2.5 is a list of three levels of models for a selected group of mathematical concepts (Wiebe 1983).

FIGURE 2.3
Three models for 25.



Model 1 Model 2 Model 3

FIGURE 2.4
A hierarchy of models for two digit subtraction with regrouping.


Two-digit subtraction with regrouping
Place-value model for 2-digit subtraction with regrouping (e.g., represent minuend on PV chart, regroup, remove subtrahend)
Non place-value model for 2-digit subtraction (e.g., represent problem with bundled sticks)
Single-object representation (e.g., represent minuend with ungrouped marbles, remove subtrahend)

Meaningful Mathematics Instruction

In the early part of this century, through the 1930s, arithmetic was taught as a large number of unrelated facts and independent skills. Little attempt was made to investigate the interrelationships between various facts and procedures, or to apply what was learned. Toward the end of the 1930s, this system of instruction was attacked because of the emergence of the child-centered movement and the growing realization that the drill method of teaching arithmetic was less than successful. Various studies showed that pupils taught arithmetic this way did not understand or enjoy the subject, they were unable to apply what they had learned to new situations, and they forgot rapidly (Trafton 1975). In the mid-1930s, the "meaning theory" became popular. The Tenth Yearbook of the National Council of Teachers of Mathematics stated that "the meaning theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes" (*The Teaching of Arithmetic* 1935, 19).

In 1947, Brownell further refined the definition of meaningful arithmetic, listing four groupings of mathematical meanings:

1. Basic concepts: meanings of whole numbers, common fractions, decimal fractions, percents, ratio and proportion.

FIGURE 2.5
Three levels of models for teaching mathematics.

Symbolic Experience Level	*Single-digit Numerals*	*Multidigit Addition*	*Single Digit Multiplication*	*Multidigit Division*	*Fractional Numerals (Ratio Meaning)*	*Division of Fractions*
Level III	Appropriate number of objects arranged in shape of numeral (e,g., 6 beans glued to numeral 6)	Place value chart addition	Repeated addition using place value chart	Division of objects on place value chart into equal groups	Arrange representation into "numerator" and "denominator" separated by line (e.g., place value chart with 7 units over p.v. chart with 1 ten and 2 units)	No known model
Level II	Cuisenaire rods	Numeration block addition	Cuisenaire-rod multiplication	Colored chip division (no place value chart)	Represent ratio with colored chips (e.g., 7 whites: 1 blue and 2 whites)	Cuisenaire rods
	Unifix cubes	Bundled stick addition	Arrays of objects (e.g., geoboard multiplication)	Cuisenaire-rod division	Cuisenaire-rod representation	Strips of paper on a numberline
Level I	Sets of single objects	Joining sets of single objects	Combining equal sets of single objects	Dividing set of single objects into equal groups	Represent ratio with single objects (e.g., 7 beans in one set: 12 beans in another set)	Partitioning portions of whole

2. Fundamental operations: knowing when (and how) to add, subtract, multiply, and divide.
3. Principles, relationships, and generalizations in arithmetic: for example, the commutative property and what happens when you add 0 to a number.
4. The decimal system: how it is used in our computational procedures and algorithms. (257)

The meaningful approach for teaching mathematics has come to be almost universally accepted by educators. Virtually all mathematics methods textbooks, textbook series for children, and articles in journals like the *Arithmetic Teacher* are based on this approach. One reason is the solid research backing that meaningful mathematics instruction has received. Weaver and Suydam did an exhaustive review of research on meaningful approaches to teaching arithmetic and concluded that

> all in all, from research conducted during the past several decades we have come to be reasonably certain that particular advantages will accrue from meaningful mathematics instruction as opposed to rote instruction. (1972, 65)

A variety of researchers have concluded, in addition, that meaning leads (1) to greater retention of what has been taught, (2) to greater transfer potential, leading to improved problem-solving ability and improved ability to apply what has been learned, and (3) to increased ability to solve new problems independently. Based on the findings of the studies they reviewed, Weaver and Suydam (1972) conclude that teachers should spend an increased amount of time using manipulative materials in discussions and explanation of mathematical concepts and procedures, and that practice should be in short specific periods, following the time spent in development of topics.

EXAMPLE: When teaching children the one-digit multiplication combinations (e.g., 4 × 6), one approach would be to have children memorize all the combinations of a single-digit number multiplied by a single-digit number (the multiplication tables) without prior explanation or discussion. A meaningful approach would be to present children with a variety of models for the concept of multiplication and for specific multiplication problems. For example, 3 times 4 could be represented by three sets with four objects in each (Figure 2.6). Another model would be to represent similar problems with arrays. For example, the problem 7 × 3 could be represented, as in Figure 2.7, with an array of dots drawn on a sheet of paper or on the chalkboard that has 7 rows and 3 columns.

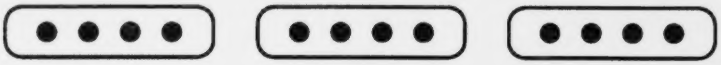


FIGURE 2.6
A set model for 3 × 4.

The teacher illustrates a variety of multiplication problems with the materials. Children are given exercises in which they represent multiplication problems with materials or pictures, or interpret these models. For example, they could be asked to write Logo programs that draw sets of intersecting lines to illustrate multiplication problems, and they would draw 3 parallel horizontal and 4 vertical lines as shown in Figure 2.8 to model 3 × 4.

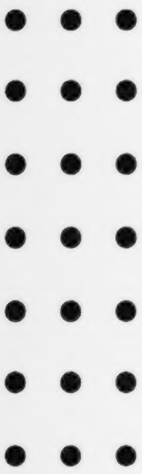


FIGURE 2.7
An array model for multiplication.

Next, the teacher could present the multiplication "table" and the class could investigate patterns in the numbers. The class could count together by 2s, 3s, 4s, and so on, to help remember and understand the table. Activities could be arranged to help children discover the fact that the order in which you multiply two numbers is irrelevant (the commutative property for multiplication). For example, the children could multiply pairs of numbers using their calculators and discuss the results.

Discovery Learning

Discovery learning occurs when the learner comes to a generalization about a concept or phenomenon for himself or herself. For example, a child, after appropriately directed activities, might realize that whenever you multiply a quantity by zero, the result is zero. Or a child may discover when writing procedures in Logo that a square has four equal sides and four interior angles of 90 degrees. As anyone who has taught geometry through Turtle Graphics knows, discovery is a very exciting way to learn. (See Chapter 7 for more information on Logo and Turtle Graphics.) The excitement of discovery in a particular area often affects learning in other areas as well, improving overall attitudes and performance.

One of the main advocates of discovery learning has been Jerome Bruner. In his influential book, *The Process of Education* (1960), he stresses the importance not only of helping children master fundamental ideas in a specific field, but also "the development of an attitude toward learning and inquiry, toward guessing and hunches, toward the possibility of solving problems on one's own" (p. 20). He points out that it is possible to structure math and science instruction in such a way that many important principles are discovered by children themselves. This contrasts with the approach where the teacher presents a generalization and examples and the students are given practice exercises. Those advocating the discovery approach to instruction claim that it promotes retention and application of what is taught, and research generally supports these contentions (Riedesel and Burns 1973).

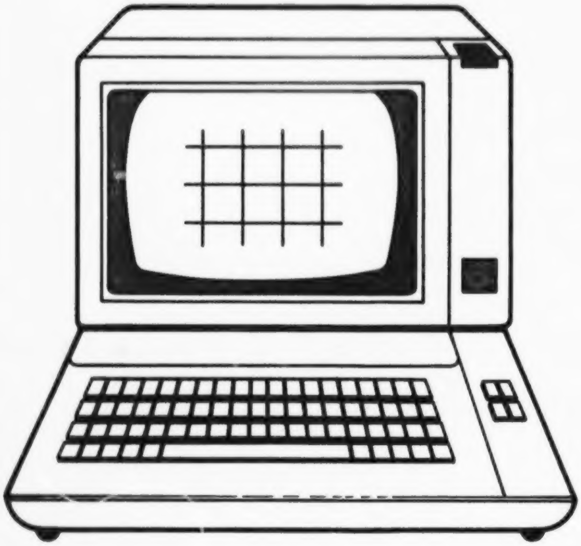


FIGURE 2.8
A logo representation for 3 × 4.

The type of discovery that takes place in the classroom is often referred to as "guided" discovery. Here, the teacher arranges a sequence of activities that lead a student to discover the desired concept. Left to their own devices, few children would ever discover the value of pi (the ratio between the circumference and the diameter of a circle). However, most children will discover this relationship after a well-designed instructional sequence. (See Figure 2.10 later in this chapter and the accompanying text for a detailed description of this activity.)

Instruction through discovery is not without problems, however. Discovery learning is usually more time-consuming than other forms of learning. Researchers have also found that failure to discover causes frustration and superstition about the problem-solving process; thus, it may not be appropriate for all children (Riedesel and Burns 1973). Furthermore, because discovery is so exciting, it is difficult for children to keep discoveries to themselves—they want to share their discoveries with other children. Therefore, only a few children in a class may actually be making discoveries while the rest are learning didactically from other children.

The emergence of the computer in the classroom may make discovery a reality for every child because with well-designed computer-assisted instruction each child can be given appropriately paced hints and helps to lead to independent discovery—something that is impossible for a teacher to do in a large-group setting. Software developers need to write and teachers need to find programs that adjust the kind and number of hints given to the child, that present graphic displays and models as needed, and that intervene and tell the child the generalization if the frustration level is reached (Wiebe 1983).

Learning Styles

The evidence is mounting that every student has his or her own innate learning style and that teachers have individual teaching styles that may or may not connect with the learning styles of students. An example is the child who is socially and group oriented. This child shuns individual achievement and success, and instead looks for guidance, approval, and reinforcement from the group. He would rather contribute to a group problem-solving activity than solve a problem individually. The teacher might be surprised and dismayed when she develops a math puzzle and offers a prize to the first pupil to complete it. When she introduces it into the class, this student may refuse to participate enthusiastically, or even "cheat" by helping a friend to win. Another example is the child who rushes through an activity, "completing" it haphazardly, and then looks for ways to amuse herself while everyone else has barely begun the task.

One aspect of cognitive style that is relevant to mathematics learning is field independence/field dependence. Analytical people (field independents) tend to look at situations as discrete parts. In mathematics, for example, they might concentrate on the steps of an algorithm rather than its overall meaning. They tend to be comfortable working by themselves in competitive situations. Global (field-dependent) people tend to look at situations as a whole and have trouble with details. They tend to feel comfortable working in noncompetitive group situations. An analytical student might feel more comfortable working on drill problems, while a global student might find open-ended problem-solving situations that require more divergent thinking to be more enjoyable (Speer 1979).

Cognitive tempo is also relevant to the mathematics classroom. Reflective people carefully and systematically examine new information and deliberate for a long time before making decisions. Such a child may do poorly on examinations or tasks because he never completes the questions. At the other extreme, impulsive individuals tend to rush through activities, not completing them satisfactorily. They may do poorly on mathematics tasks because they do not properly think them through and make many mistakes (Kagen 1965). Presentations by teachers to the average child in the classroom will be too fast for the reflective child and too slow and boring for the impulsive child.

Invention

An important contribution of recent research in the development of mathematical understanding in young children is that instruction is not the sole determinant of what children learn. It appears that children actively construct knowledge for themselves through interaction with the environment and the reorganization of their own mental constructs (Steffe, vonGlaserfeld, Richards, & Cobb 1983). When children are confronted with a learning situation, they do not simply absorb the information. Instead they change it to fit into their own mental framework. Indeed, many studies are showing that children invent much of their own mathematics (Resnick 1976) and that they come to school with a highly developed personal mathematical system (Romberg and Carpenter 1986, 853).

An example of the self-developed techniques young children use to solve mathematics problems is the different strategies they use when adding numbers less than ten. The most primitive method is to count from one up to the first addend, then count on (using fingers or objects) according to the number in the second addend (e.g., in solving "2 + 7" the child counts "One, two," then using fingers, counts on seven more to obtain nine). Other self-developed strategies are to start counting after the left-most addend (e.g., in the problem "4 + 6" the child counts six units beyond four) or to start counting after the largest of the addends (e.g., in the problem "2 + 7" the child counts two beyond seven).

According to Romberg and Carpenter (1986), research on children's mathematical invention can help guide early instruction in mathematics. We as teachers need to capitalize on children's natural problem-solving abilities, using word problems as a basis for developing mathematical concepts. We should not jump too quickly from the representation of operations on numbers with concrete materials to memorization of math facts. We need to recognize that children use strategies like *counting on* and *counting back* to solve addition and subtraction problems for an extended period of time and that this is a natural part of their development.

TEACHING MATHEMATICS

The theoretical considerations and research findings discussed in the first part of this chapter lead to several suggestions for the teaching of mathematics. In order to produce the greatest amount of learning in children, teachers need to teach for understanding, use models, help students understand algorithms, use discovery techniques when possible, teach for application and problem solving, make use of varied media and materials, use drill at the appropriate time, and make mathematics interesting and applicable.

Develop Understanding

Two students I worked with recently had difficulty remembering certain multiplication facts. Jenny had a terrible time remembering 9 × 6, 9 × 7, and 9 × 8. June could never seem to recall the answers to 8 × 6 and 8 × 7 and several of the products involving nines. When June forgot the products, she would simply guess at an answer. However, when Jenny forgot a product, she would eventually come up with the answer. Sometimes she drew intersecting sets of parallel lines and counted the intersections. Other times, she worked from products she knew and added until she came to the forgotten product. For example, if she forgot 9 × 7, she would start at 45 (9 × 5), then would add two more nines.

Although certain children might be relatively successful doing pencil and paper computations without much understanding of what they are doing, they are in trouble when they forget the steps in the computational algorithms or their "facts" (sums and products of single-digit numbers—3 + 8 or 5 × 4, differences with single-digit differences and subtrahends—12 − 7, and quotients with single-digit quotients and divisors—72 ÷ 9). Moreover, they often have difficulty solving problems and applying mathematics. On the other hand, students who understand the fundamentals of mathematics are often able to use this knowledge to help them where their memory fails, or in new applied and problem situations. For example, when asked to estimate the sum of $^{15}/_{16}$ inches and $^{7}/_{8}$ inches in an applied measurement problem, a child who does not understand fractions would probably try to apply the algorithm for adding whole numbers to the adding of the fractions whereas a child who understands fractions would round each to one inch and add to get an approximate answer.

Developing understanding of mathematics involves teaching children what the basic operations mean by presenting real-life examples of these operations, by using instructional aids such as models, and by relating the new concept to a previously mastered concept. For the teacher, it involves giving students a thorough acquaintance with our numeration system—whole numbers, fractions, decimals, and percentages. For the student, it involves developing an understanding of the fundamental unifying principles of mathematics and consequently an understanding of *how* and *why* computational algorithms work.

Use Models

Some of the basic models for teaching mathematics are presented below. Many of these are easily made by the teacher. Others can be purchased from school-supply businesses. The list is not complete—for example, a variety of similar materials (under different names) can be used to teach numeration and whole-number operations. It does, however, represent the major models needed to teach mathematics.

Model	*Topics Taught*
Bundled Sticks	Numeration, Whole-number operations
Numeration Blocks	Numeration, Whole-number operations, Measurement, Metric system, Decimals
Bean Sticks	Numeration, Whole-number operations
Colored Chips	Numeration, Whole-number operations, Bases other than ten
Place Value Charts	Numeration, Whole-number operations, Decimal numeration, Decimal operations

Number Lines	Whole-number operations, Integers, Operations on fractions, Operations on decimals, Operations on reals
Grids	Fractions, Decimals, Percentages
Elastic	Operations on Percentages
Geoboard	Geometry, Measurement, Fractions

When using models or any other instructional device, be cognizant of the individual needs of children. The number and types of materials needed to help a given child understand a mathematical concept vary greatly. Certain children may be able to understand abstract ideas like place-value without any model. Others may need a series of models, from simple to abstract, to help them understand. Remember, models are tools to help promote understanding. Use them wherever and as long as needed, then move to the symbolic level.

Teach Relevant Computational Algorithms

An algorithm is a sequence of steps for finding answers to a particular class of problems. An algorithm for killing a pesky fly is (1) get a flyswatter, (2) wait until the fly has settled on a non-breakable object, and (3) swat it with the fly swatter. (4) If you miss, return to step 2. There are many other possible algorithms for killing a fly such as spraying the room with insect spray, closing off the room where it is until it starves to death, shooting it with a shotgun, or closing it in the house and burning the house down. Obviously, some algorithms for killing the fly are better for some people than others. For example, a person with bad eyesight and slow reflexes would probably be better off using an insect fogger than a fly swatter.

In arithmetic, the algorithm we usually use for multiplying 16 by 8 is to (1) write the two numbers in vertical form, (2) multiply the 6 by the 8, (3) "carry" the 4 of the answer to the tens column, (4) multiply the 1 in the tens column by the 8, (5) add the 4 to the 8, (6) write the 12 (120), and (7) the resulting numeral (128) is the product. There is nothing sacred about this algorithm; many others could do the job:

a. Make an 8 by 16 array of objects and count the total number of objects.
b. Add eight 16s.
c. Enter 8 × 16 = in your calculator.
d. Enter and run the following program on a computer.

```
10 DATA 8,16
20 READ A, B
30 LET C = A * B
40 PRINT ``8 TIMES 16 EQUALS''; C
50 END
```

e. Write each "partial product" below the line and add.

$$\begin{array}{r} 16 \\ \underline{\times\ 8} \\ 48 \\ \underline{80} \\ 128 \end{array}$$

The algorithm we select for resolving a particular situation depends on the nature of the situation. When balancing their checkbook or computing income taxes, most people use a calculator. When computing 15% of a restaurant bill, most people estimate and/or mentally find 10% by moving the decimal to the left one place, take half of that value and add the two values together. Although I learned in eighth grade to compute square roots using pencil and paper, that technique has long since been forgotten and I either use an estimation technique or a calculator.

In the work place, everyone who must do calculations with any regularity has a calculator, computer, or cash register to do the computations for them. In other situations, estimates are often sufficient. Occasionally, we need to do pencil and paper computations. The question we must ask ourselves is, "Is the amount of time spent teaching a particular pencil and paper computational technique relative to its importance in everyday, adult life?"

Recognizing that pencil and paper algorithms will occasionally be useful in their later life and, more importantly, that pencil and paper computational strategies can be used to develop understanding, we should teach children a variety of computational techniques, first introducing inefficient but meaningful algorithms, then moving to more efficient techniques. The following presents two pencil-and-paper methods for multiplying 25 by 5.

$$\begin{array}{r} {}^{2}25 \\ \underline{\times 5} \\ 125 \end{array} \qquad \begin{array}{r} 25 \\ 25 \\ 25 \\ 25 \\ \underline{+25} \\ 125 \end{array}$$

The first is efficient, but has many steps to remember, some of which have abstract meanings (e.g., carrying). It is based on a more complex principle than the second (the distributive property). The second is inefficient, but easy to remember—all you need to remember is that you keep adding 25 until you have added 5 of them. It also helps the student realize that one meaning of multiplication is repeated addition.

The upper level of efficiency depends on the learner; some students may never move beyond the most inefficient methods for computing, such as doing repeated addition for multiplication, or repeated subtraction for division. These students, of course, will be dependent upon calculators during the rest of their lives for most computations. Some decry this approach, saying that we are making students slaves of technology. In response, the present method of drill, drill, drill on computational techniques that certain students will never be able to master is robbing many students of any functional ability in arithmetic. Not only do these students fail to learn the computations we attempt to drill into them, but they also are deprived of the mathematical meanings they need to function in society. They also fail to learn the techniques that are within their capabilities, including the use of the technological tools like calculators, that will enable them to function in our quantitative society.

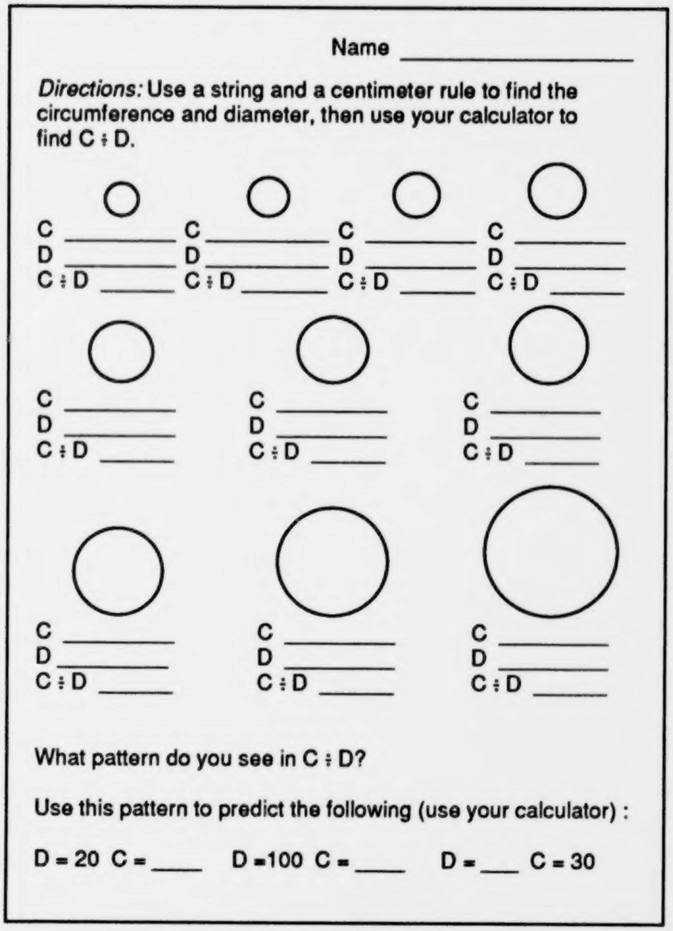
Teach through Discovery

When a child discovers something for himself, chances are that he will remember it longer and be able to use it in more settings than if it is simply told to him. Discovery occurs when the student is not told a concept or technique in final form but comes up with the principle and/or steps leading to the principle himself. Perceptive, creative teachers find ways to incorporate discovery into many lessons, even when those lessons must be highly structured.

The following are two examples of discovery teaching. The alternative to these techniques is to tell students the principle (e.g., the Pythagorean theorem, or the commutative property).

EXAMPLE 1: Students are given a worksheet with a sequence of circles whose diameters are in whole centimeters (exactly 6 cm, 9 cm, and so forth). They use string and a ruler to find the diameter and circumference of each circle. They record the diameters and circumferences and use a calculator to divide each circumference by its diameter. They then attempt to state the relationship between these two values. (See Figure 2.9.)

FIGURE 2.9
A worksheet for discovering pi.



Name ________________

Directions: Use a string and a centimeter rule to find the circumference and diameter, then use your calculator to find C ÷ D.

C ________ C ________ C ________ C ________
D ________ D ________ D ________ D ________
C ÷ D _____ C ÷ D _____ C ÷ D _____ C ÷ D _____

C ________ C ________ C ________
D ________ D ________ D ________
C ÷ D _____ C ÷ D _____ C ÷ D _____

C ________ C ________ C ________
D ________ D ________ D ________
C ÷ D _____ C ÷ D _____ C ÷ D _____

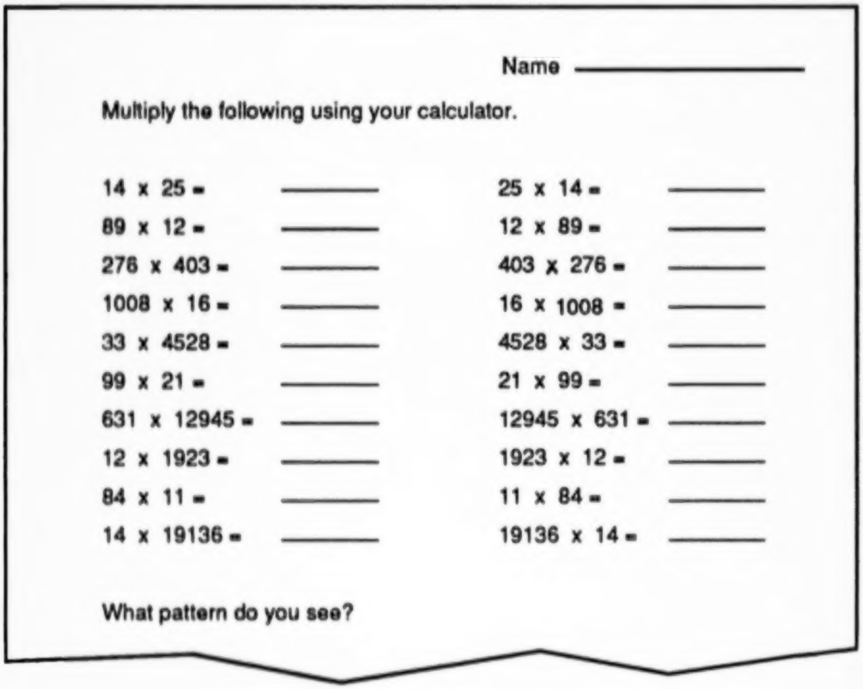
What pattern do you see in C ÷ D?

Use this pattern to predict the following (use your calculator) :

D = 20 C = ____ D = 100 C = ____ D = ____ C = 30

EXAMPLE 2: Students multiply sequences of numbers given on a worksheet using their calculators, and record the answers. They investigate the number pairs to see if there are any patterns. Next, they state the generalization that they discover. (See Figure 2.10.)

FIGURE 2.10
A worksheet for discovering the commutative property.



Name ______________

Multiply the following using your calculator.

14 x 25 = ______ 25 x 14 = ______
89 x 12 = ______ 12 x 89 = ______
276 x 403 = ______ 403 x 276 = ______
1008 x 16 = ______ 16 x 1008 = ______
33 x 4528 = ______ 4528 x 33 = ______
99 x 21 = ______ 21 x 99 = ______
631 x 12945 = ______ 12945 x 631 = ______
12 x 1923 = ______ 1923 x 12 = ______
84 x 11 = ______ 11 x 84 = ______
14 x 19136 = ______ 19136 x 14 = ______

What pattern do you see?

Teach Estimation Skills and Mental Arithmetic

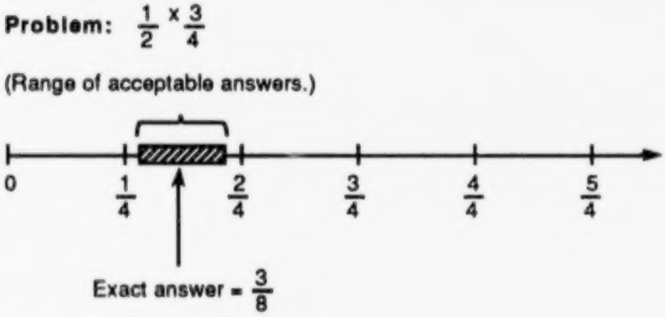
Even in the days when pencil and paper were king, the ability to estimate was extremely important. People needed to know if the answers to their computations were accurate and they needed to mentally determine approximate answers to many applied problems.

Estimation ability grows partly out of mathematical understanding and experience, but more importantly it can be elicited by the effort of teachers. Estimation and mental computation ability should be explicitly taught. On a regular basis, teachers should ask students to think of and to write down the answers to problems they put on the board (both computational and word problems) without doing any pencil and paper computation. Teachers should then give students the range of acceptable answers and discuss strategies for obtaining those answers. Children in the class whose answers are acceptable might be asked to share their technique for finding the answer. Such lessons should be followed by drill activities where students write their estimate to problems before they do any computations, then use manipulatives, calculators, pencil and paper, etc. to find the precise answer and evaluate the appropriateness of their estimate.

EXAMPLE: The teacher places the problem "$\frac{1}{2} \times \frac{3}{4}$" on the board and asks children to write what they think the answer is without doing any computation. She then asks several children what his/her estimate was. She asks one or two whose answers are within a reasonable distance of the precise answer to explain how they came up with their answers. She then places a number line (marked in fourths) on the overhead and marks a range of the line in red that is near half of $^3/_4$. Children are then asked to name fractions in the marked range. To illustrate, the teacher places overlays on top of the original that are marked in thirds, fifths, etc. Each time she asks the students to name fractions that are in the marked range. (See Figure 2.11.)

Note that Chapter 6 covers the topic of computational estimation in detail.

FIGURE 2.11
Estimating fractions.


Problem: $\frac{1}{2} \times \frac{3}{4}$
(Range of acceptable answers.)
0
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
$\frac{5}{4}$
Exact answer = $\frac{3}{8}$

Teach for Problem Solving

As stated in Chapter 1, problem-solving proficiency should be the final goal of mathematics instruction. Problem-solving ability is the net product of all the teaching strategies discussed in this chapter. Children become good problem solvers when they understand mathematics, when they know general and specific mathematics principles and skills, when they have a variety of computational strategies at their fingertips, and when they know how to use technology. In addition, teachers can help children become better problem solvers by explicitly teaching problem-solving strategies and by having students solve a variety of routine and nonroutine problems. Chapter 5 discusses methods for teaching problem solving.

Make Use of a Variety of Materials and Media

When childrens' lives outside the classroom are filled with television, computers, audio equipment, movies, and multimedia rock music, a classroom that uses only textbooks, chalkboards, and pencil and paper will be a dull place. A classroom where math is taught should be filled with a variety of materials and media. The teacher should use films, television shows, newspapers, advertisements from department and toy stores, cassette tapes, film strips, overhead transparencies, slides, flannel boards, and large manipulatives for demonstration.

The classroom should also have a place where children can go to get needed manipulatives, reference books, construction materials, worksheets containing problems, and so on. This "math lab" or "math corner" should be a place where children can find what they need to solve mathematical problems or to explore the world of mathematics. Chapter 19 discusses setting up and managing a math lab.

Use Appropriate Drill and Practice

Many people have had the experience of learning to play tennis or golf without proper lessons. When this happens they often practice the bad habits they learn, thus reinforcing them. Unfortunately, this can happen when children learn mathematics if they are asked to practice things they have not understood. On the other hand, boredom and frustration may result when a student is asked to practice something she already understands and can do without difficulty. Chapter 20 gives suggestions for making drill interesting and relevant.

SUGGESTED READINGS

Adler, I. Mental growth and the art of teaching. *Arithmetic Teacher* 13 (7), March 1966: 576–584.

Driscoll, M. What research says. *Arithmetic Teacher* 31 (6), February 1984: 34–35.

Elkind, D. What does Piaget say to the teacher? *Today's Education* 61 (8), 1972: 46–48.

Fennema, E. Manipulatives in the classroom. *Arithmetic Teacher* 20 (5), January 1973: 350–352.

Gagne, R. M. *The Conditions of Learning*. New York: Holt, Rinehart and Winston, 1977.

Gibb, E. G. Teaching effectively and efficiently. *Arithmetic Teacher* 29 (7), March 1982: 2.

Greenwood, J., and Anderson, R. Some thoughts on teaching and learning mathematics. *Arithmetic Teacher* 31 (3), November 1983: 42–49.

Hamrick, K. B. Are we introducing mathematics symbols too soon? *Arithmetic Teacher* 28 (3), November 1980: 14–15.

Hiebert, J. Why do some children have trouble learning measurement concepts? *Arithmetic Teacher* 31 (7), March 1984: 19–24.

Masalki, W. J. Mathematics and the active learning approach. *Arithmetic Teacher* 26 (1), September 1978: 10–12.

Nasca, D. Math concepts in the learner centered room. *Arithmetic Teacher* 26 (4), December 1978: 48–52.

Wiebe, J. H. Physical models for symbolic representations in arithmetic. *School Science and Mathematics* 83(6), October 1983: 492–502.

QUESTIONS

1. How are models used in subject areas other than mathematics, such as science, sociology, and architecture?
2. Discuss how you could make a model for teaching whole-number numeration (units and tens only) to children that is similar to bundled sticks but using dried pinto beans and small plastic pill bottles. How could you extend the model to include hundreds?
3. In groups of 3 or 4, discuss some mathematical ideas that you discovered yourselves. Discuss how these ideas can be presented in school so that children discover them for themselves as you did.
4. Discuss a concept or procedure that you practiced extensively without understanding when you were in elementary school. When and how did you come to understand the concept? What impact did this have on your achievement in that area?

5. What estimation techniques do you use that you developed yourself? Would they be good techniques to teach children? Why or why not?
6. Discuss a unit of instruction (three lessons) about decimal numerals (the meaning of ".05", "1.25", etc.) that makes use of the following: models, television, calculators, and audio tapes.
7. Develop a worksheet that is designed to lead children to discover that 1 is the identity for multiplication.
8. Discuss a learning activity using calculators that is designed to develop estimation skills involving division.

CHAPTER

3

The Calculator in the Mathematics Classroom

When the pocket electronic calculator first appeared over a decade ago, cries of anguish could be heard from a variety of sources including many parents, the media, and even some teachers. They feared that this new device would trigger a major decline in mathematics skills. They pictured the younger generation asking, "Why learn math when we have machines to do it for us?" Subsequently, without any serious attempt to determine the exact effect calculators might have on students who use them regularly or on the types of things schools should teach in the electronics age, many parents and school administrators tried to keep calculators completely out of the classroom, at least until children had learned their pencil and paper computational algorithms.

Although such fears still persist among some uninformed parents and educators, they have been shown to be groundless. Calculator usage in the classroom has been heavily researched. According to Marilyn Suydam (1980) and Roberts (1980), the results of these studies have shown almost unanimously that *calculators do not harm student achievement*. In fact, the majority of these studies have shown quite the opposite—calculators tend to improve student understanding of mathematics, their pencil and paper computational skills, and their scores on achievement tests even when they are not allowed to use calculators on the tests after having been taught with them. Calculators also enhance students' attitudes toward mathematics, their problem-solving performance, and their creativity when attacking problems. Many researchers and teachers also report a renewed interest in and excitement about mathematics when calculators are used. Indeed, the experience of researchers and thousands of teachers has given us plenty of ammunition for following the admonition of the National Council of Teachers of Mathematics that "mathematics programs must take full advantage of the power of calculators . . . at all grade levels" (*An Agenda for Action* . . . 1980).

This chapter looks at calculator use in the mathematics classroom and its impact on the mathematics curriculum. It also gives suggestions for selecting the kind of calculator that is appropriate for the elementary mathematics classroom. Many of the suggestions and discussions initiated in this chapter are expanded in later sections of the book.

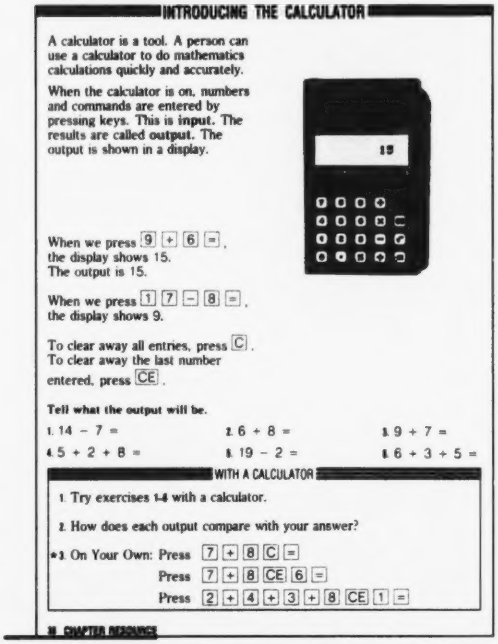
TEACHING MATHEMATICS WITH CALCULATORS

Now that we can agree that calculators can be good for students and should be fully incorporated into the classroom, the question is, "How"? Calculator usage should go much beyond checking student's answers after they have solved exercises using pencil and paper, though this use is certainly valid. They should be used (1) in developing understanding of the principles of mathematics, (2) in problem solving and applied mathematics activities, (3) as a regular part of drill and practice.

Developing Mathematical Understanding with Calculators

Several features of calculators make them valuable for use in developing mathematical understanding.

- Because of the speed and accuracy of calculators, students can do numerous computations in a short period of time, allowing them to do tedious, but meaningful algorithms that are usually bypassed for more efficient ones.
- The computational speed of the calculator also allows students to do directed, repeated estimations followed by checking—a procedure that can help develop understanding of numerous concepts and procedures as well as students' estimation ability.
- In addition, most calculators have a constant feature that can be used to "count," to do repeated addition and subtraction, and to find powers.
- Finally, calculators work as quickly and easily with large numbers as with small numbers allowing their use in realistic problem solving activities.



INTRODUCING THE CALCULATOR

A calculator is a tool. A person can use a calculator to do mathematics calculations quickly and accurately.

When the calculator is on, numbers and commands are entered by pressing keys. This is **input**. The results are called **output**. The output is shown in a display.

When we press $\boxed{9}$ $\boxed{+}$ $\boxed{6}$ $\boxed{=}$, the display shows 15. The output is 15.

When we press $\boxed{1}$ $\boxed{7}$ $\boxed{-}$ $\boxed{8}$ $\boxed{=}$, the display shows 9.

To clear away all entries, press $\boxed{C}$. To clear away the last number entered, press $\boxed{CE}$.

Tell what the output will be.

1. 14 − 7 =
2. 6 + 8 =
3. 9 + 7 =
4. 5 + 2 + 8 =
5. 19 − 2 =
6. 6 + 3 + 5 =

WITH A CALCULATOR

1. Try exercises 1-6 with a calculator.
2. How does each output compare with your answer?

★3. On Your Own: Press $\boxed{7}$ $\boxed{+}$ $\boxed{8}$ $\boxed{C}$ $\boxed{=}$
Press $\boxed{7}$ $\boxed{+}$ $\boxed{8}$ $\boxed{CE}$ $\boxed{6}$ $\boxed{=}$
Press $\boxed{2}$ $\boxed{+}$ $\boxed{4}$ $\boxed{+}$ $\boxed{3}$ $\boxed{+}$ $\boxed{8}$ $\boxed{CE}$ $\boxed{1}$ $\boxed{=}$

38 CHAPTER RESOURCES

From the Silver Burdett & Ginn Mathematics Series, Grade 3. Copyright 1987 by Silver Burdett & Ginn, Morristown, NJ. Used with permission.

BEST COPY AVAILABLE

(I) Using the Constant Function

Different brands of calculators have different methods of activating the constant function. Many inexpensive calculators will automatically repeat the last operation entered each time you press the = button if no other number or sign has been entered since the last = was pushed. To see if your calculator works this way, press

1 + 1 = = = = etc.

Each time you press the = button, the " + 1" operation is carried out and the next higher counting number is displayed.

With other inexpensive calculators, the constant function is activated by pushing an operation button twice. To enter a constant, first push the number that you wish to repeatedly operate on. Next, press the operation button twice:

2 + +

When you do this, many calculators will display a "k" or a "c" for *constant*. Once the constant has been entered, you may use it by pressing the number you are going to add (subtract, multiply, or divide) the constant to, and hitting the equal sign. Suppose you own a store and are having a post-Christmas sale with everything in the store reduced by 25%. To figure this reduction on your calculator, you would first enter the constant.

.75 × ×

Then you would enter the price of each item and press the equal button. For example,

24.95 = (18.7125 is displayed.)
36.45 = (27.3375 is displayed.)

To use this feature to count, enter the constant, then enter the number from which you are starting to count, then repeatedly press the = button. For example, if you wanted to count backwards from 100 by fives, enter

5 − − 100 = = = etc.

Other calculators have a constant button or other techniques for implementing the constant feature. Consult the owner's booklet for instructions for using these other types of constants.

Counting techniques are used as a way to develop understanding of multiplication and division: one meaning of multiplication is repeated addition and one meaning of division is repeated subtraction. Often times these meanings are introduced with single-digit numerals, then forgotten in later grades because of the difficulty of doing them with pencil and paper. Children's errors when doing pencil and paper computations with larger numbers demonstrate that they have not transferred the meaning they learned with single-digit operations to larger numbers and that they tend to attack larger problems in a rote, piecemeal fashion. Following are some examples of how the constant function can be used to develop understanding of multidigit multiplication and division. Chapters 8 through 14 have background information on these examples and other examples of this technique:

EXAMPLE 1: 125 × 25 = ?

SOLUTION: 125 + 125 = = = = = (24 times)

SOLUTION: 125 + + = = = = = (24 times)

(Note that the first time you press " = ", you have already added two 125s. The two solutions shown here are for the two types of constant function discussed above.)

EXAMPLE 2: 2408 ÷ 43 = ?

SOLUTION: 2408 − 43 = = = = …

SOLUTION: 43 − − 2408 = = = …

(until the display reads "0").

Count the number of times the = button is pushed.

EXAMPLE 3: 69 × 23 1/3 = ?

SOLUTION: 69 + 69 = = = (23 times)
M+ 69 × 1 − 3 = M+ MR

EXAMPLE 4: 453 ÷ 24 = ?

SOLUTION: Subtract 24 from 453 until you obtain a display that is less than 24 but greater than 0. Express the display either as a remainder or divide by 24 to convert to a decimal.

(2) Directed, Repeated Estimation

Because the calculator allows the user to make a large number of computations in a short period of time, students can be asked to find answers to computational problems using repeated estimation. This technique can give them practice in estimation and problem solving and, in many cases, will help develop understanding of computational processes.

To develop understanding of multidigit division, represent the division problems as multiplication problems with a missing factor. Have the students estimate the missing factor, multiply, estimate again, etc., until they obtain a correct answer. Later the problems can be written as usual division problems but solved through the same estimation technique so that students see the relationship between division and multiplication and so that they become accustomed to using estimation to determine the reasonableness of answers obtained in pencil and paper computations.

EXAMPLE 5: Estimate the missing factor. Multiply on your calculator to see how close your estimate is. See if you can find the answer in 8 steps or less.

Answer	*Estimate*	*Answer*
A. 43 × ______ = 2408	(1) ______	______
	(2) ______	______
	and so on.	
B. 12 × ______ = 672	(1) ______	______
	(2) ______	______
	and so on.	

EXAMPLE 6: Estimate the quotient. Multiply by the divisor to see how close your estimate is. See if you can find the answer in 10 steps or less.

Answer		*Estimate*	*Answer*
A. $16\overline{)5008}$	(1) 16 × =	______	______
	(2) 16 × =	______	______
	and so on.		

To develop understanding of the relationship between addition and subtraction, subtraction problems could be presented as addition problems with a missing addend, similarly to the problems above. Students estimate the missing addend, then check to see how close their estimate is.

The meaning of square roots can be demonstrated by having students estimate the square roots of various numbers, then by multiplying the numbers by themselves on the calculator to check the answers. Using repeated trial with estimation, they will discover it is possible to find the square root of irrational numbers to several decimal places quite quickly. Not only is this a useful technique for times when they do not have a calculator with a square root button, but it also develops their understanding of the principle.

To find the square root of an irrational number, try finding two consecutive numbers whose squares are larger and smaller than the desired number, progressing from whole numbers to numbers with tenths, to numbers with hundredths, etc. The following sequence shows one way to arrive at the square root of 10 to the nearest 100th.

EXAMPLE 7: $\sqrt{10} = ?$

	Estimate	*Trial*	
STEP 1:	5	5 × = (25)	
STEP 2:	4	4 × = (16)	
			($\sqrt{10}$ is between 3 and 4 because 10 is between 3^2 and 4^2)
STEP 3:	3	3 × = (9)	

Since 10 is close to 9, one might estimate that $\sqrt{10}$ is close to 3.

	Estimate	*Trial*		
STEP 4:	3.1	3.1 × =	(9.61)	$\sqrt{10}$ is between 3.1 and 3.2
STEP 5:	3.2	3.2 × =	(10.24)	
STEP 6:	3.15	3.15 × =	(9.9225)	
STEP 7:	3.16	3.16 × =	(9.9856)	$\sqrt{10}$ is between 3.16 and 3.17
STEP 8:	3.17	3.17 × =	(10.0489)	

This process could be carried out for several more decimal places in less time than it would take to use the square root algorithm frequently taught in the eighth grade.

(3) Entering the Problem in Meaningful Pieces

Many computational problems can be made more meaningful to children by having them specify and enter substeps rather than combining or simplifying steps as we do in standard computational algorithms. For example, when adding 345 and 197 using pencil and paper, we combine *regrouping* and *adding* in the ten's place. Multidigit addition is made more meaningful by adding and recording ones, tens, and hundreds, then adding the partial sums. With the calculator, students can do these partial computations mentally, then use the calculator to quickly combine the pieces.

EXAMPLE 8: Find each partial product mentally and record it. Add them up using your calculator. Check your answer by multiplying the two numbers with your calculator the usual way.

(1)	145	(2)	339	(3)	107	(4)	830
	×67		×34		×92		×19
	35		___		___		___
	280		___		___		___
	700		___		___		___
	300		___		___		___
	2400		___		___		___
	6000		___		___		___
	9715						

Meaning for borrowing when subtracting multidigit numbers can be promoted by subtracting one place at a time and investigating changes in the place from which the value was subtracted and in the place immediately to the left.

EXAMPLE 9:	147	147	−	9	=	(Answer: 138. Note change in ten's place)
	−29	138	−	20	=	___

Perhaps the most important consideration when using the calculator in the last two ways discussed above is making sure that students actually go through the desired meaningful steps rather than simply using the quickest, but nonmeaningful, method for finding a given solution. Probably the best method for assuring this is having students record estimates or intermediate steps on worksheets as in Examples 5–8.

Problem Solving and Mathematical Applications with Calculators

When children are presented with mathematical problems (see Chapter 5 for a discussion of problem solving), they should first of all try to understand what the problems are asking them to do. They then need to develop a plan of attack and carry out their strategies. Finally, they need to look at the answer they obtained to see if it is reasonable. If not, they need to try another approach.

Where does the calculator fit into this sequence? Since the purpose of problem-solving activities is to help children develop the ability to solve problems as they occur in real life rather than to develop computational facility, the act of computing an answer should be a minor consideration. More important is understanding the problem and correctly setting up a strategy for obtaining a solution. Research has shown that children who use calculators in problem solving have more "bright ideas" and use a greater number of strategies than those who use pencil and paper computation (Wheatly 1980). Children who use pencil and paper computation tend to try only one strategy and quit—often without even checking to see if the answer they got was reasonable. Although I have no research on which to base this conjecture, I suspect that some children short-circuit the problem solving process completely by doing whatever computational strategy is easiest using the numbers in the problem (e.g., if the numbers in the problem are 48 and 2, the easiest operation is probably to add to get 50) rather than the one that is most reasonable.

When you teach problem solving using calculators, have students use appropriate problem-solving strategies. (See Chapter 5.) On occasion, you may wish to have students write the steps they use with the calculator (even those they try, then abandon) so that you may evaluate their progress toward more efficient and appropriate strategies. Students' notes may be used in lieu of exams for diagnosing thought processes and for planning instruction.

The following calculator problem solving chart (Figure 3.1) can be used in the beginning stages to help you direct students' work when they solve problems using calculators. Figure 3.2 shows a sample of a child's work using the organizational sheet. After children have solved a number of problems using the organizational sheet, give them problems written in standard form and have them write the steps they use on a blank sheet of paper.

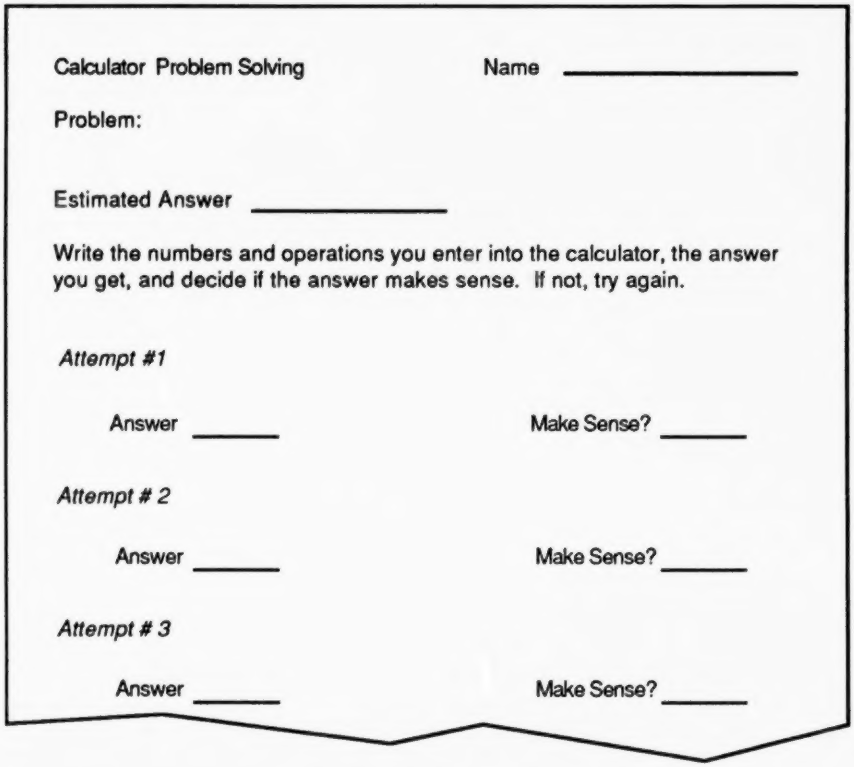


FIGURE 3.1
A worksheet for calculator problem solving.

Calculator Problem Solving Name ____________

Problem:

Estimated Answer ____________

Write the numbers and operations you enter into the calculator, the answer you get, and decide if the answer makes sense. If not, try again.

Attempt #1

Answer ______ Make Sense? ______

Attempt # 2

Answer ______ Make Sense? ______

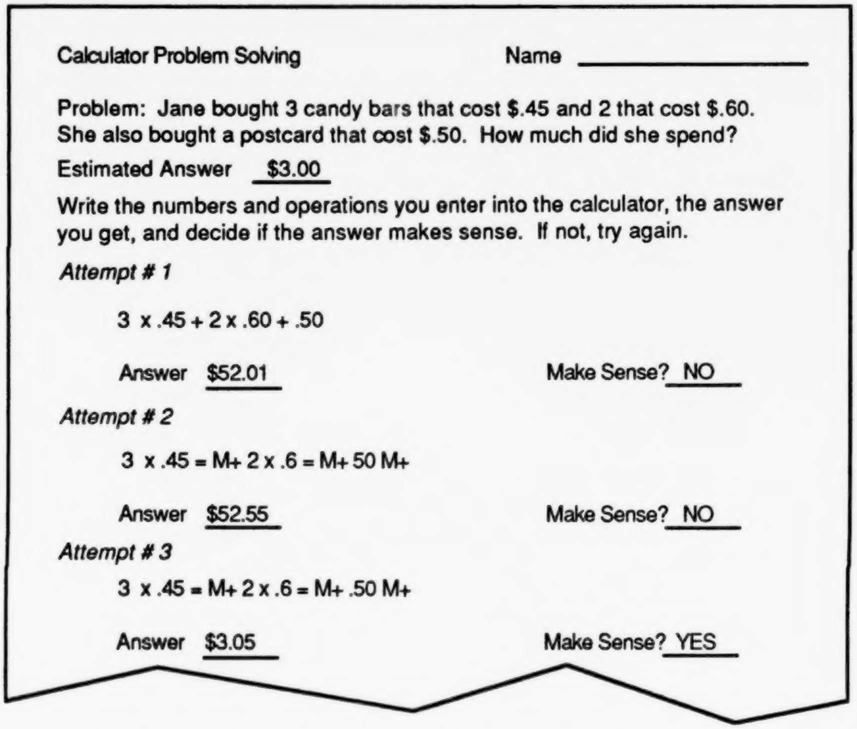
Attempt # 3

Answer ______ Make Sense? ______

Using the Calculator in Drill and Practice

The most obvious way to use the calculator in drill and practice is to first do computations using pencil and paper, then check the answer using the calculator. This provides the immediate reinforcement often lacking in drill situations. Further

FIGURE 3.2
A sample of a child's work using the calculator problem solving worksheet.



Calculator Problem Solving Name ________________

Problem: Jane bought 3 candy bars that cost $.45 and 2 that cost $.60. She also bought a postcard that cost $.50. How much did she spend?

Estimated Answer $3.00

Write the numbers and operations you enter into the calculator, the answer you get, and decide if the answer makes sense. If not, try again.

Attempt # 1

3 x .45 + 2 x .60 + .50

Answer $52.01 Make Sense? NO

Attempt # 2

3 x .45 = M+ 2 x .6 = M+ 50 M+

Answer $52.55 Make Sense? NO

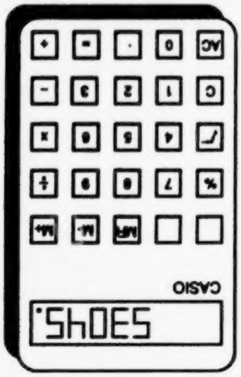
Attempt # 3

3 x .45 = M+ 2 x .6 = M+ .50 M+

Answer $3.05 Make Sense? YES

learning may be provided for older students by asking them to write what they did wrong where they made errors.

FIGURE 3.3
A "talking calculator" example.



The "talking calculator" is an extension of using the calculator to check pencil and paper computations. When the calculator is turned upside down, most numbers resemble one of the letters of the alphabet. The "calculator alphabet" follows:

Number:	1	3	4	5	6	7	8	0
Letter:	I	E	h	S	g	L	B	O

A surprising number of words can be made with this combination of letters. To use them, select problems that result in numbers that, when turned upside down, spell words. (See Figure 3.3.) Have children work the problems using pencil and paper, write the answer, enter the answer into the calculator, then turn it upside down to see the resulting word. To make worksheets interesting, have the words be answers to silly or interesting questions, or arrange them in crossword puzzles.

EXAMPLE 1: *Where do Eskimos live?* 205,843 + 294,918 = ?

EXAMPLE 2: *What can you do about your age when you grow older?*

63.4 × 5 = ?

MATHEMATICS CURRICULUM AND THE CALCULATOR

In Chapter 1, we introduced an issue even more important than whether, or how, calculators should be used in the elementary mathematics classroom. This issue is what topics to include in mathematics instruction. Should we teach long division using pencil and paper when most people use calculators to solve multidigit division problems? Shouldn't we place more emphasis on numeration and place value since estimation and mental arithmetic make heavy use of this knowledge and since estimation of results obtained from calculator computations is an important skill? As teachers, we need to be able to determine which topics included in textbooks and on outdated lists of skills should be downplayed and, on the other hand, which need to be stressed or even introduced into the curriculum. The following is a discussion of the changes calculators are and will be making in mathematics curriculum.

In today's technological society, people use calculators for most mathematics tasks, such as balancing a checkbook, computing income taxes, or calculating grades at the end of a semester. On the other hand, they do mental computations for such other tasks as determining if they have enough money in their purse to buy two sale items in the department store, what the correct amount of money is to leave as a tip in a restaurant, or if they have entered the numbers correctly when using their calculator. Thus pencil and paper computations are being squeezed at both ends—at the upper end as the calculator is used instead of pencil and paper and at the lower end as people use mental arithmetic for simpler computations or to estimate more complex computations. Because of these realities, we need to teach the following in the elementary mathematics classroom.

Place Value and Numeration

This topic has been historically underemphasized in our curriculum. Research by the author and others (Engelhardt and Wiebe 1981) has shown that understanding of our number system is fundamental to understanding computations in arithmetic. When doing mental arithmetic, it is even more important for us to know the various meanings related to numbers. (See Chapter 9 for a discussion of number concepts.) Most pencil and paper algorithms involve lining up the places in the numbers from right to left, then mechanically carrying out a sequence of operations on single-digit operands. Students may successfully carry out these algorithms without being aware of the place values of the digits in the numbers. Many of the strategies for mental arithmetic, however, focus on the front digits of the numbers and require attention to the place of these digits. For example, when estimating the sum of 4987 and 345, you need to be aware that the first digits in the two numbers have different place values. Estimation and mental arithmetic will be discussed in Chapter 6.

Basic Facts

Our schools should continue to place strong emphasis on addition, subtraction, multiplication, and division involving single-digit numbers. Since mental computation and estimation rely heavily on these facts, they should be memorized by all children.

In the age of technology, we need to append our set of "basic facts" to include the decimal equivalents of common fractions. Since neither calculators nor computers handle fractions well, the ability to mentally convert commonly-appearing fractions (e.g., those relating to measurement) to decimals will add to the student's functional mathematical literacy. Note that while it is possible to convert fractions to decimals on a calculator and then carry out the needed operations, it is not usually desirable. Compare the keystrokes of the problem *2 ⅛ inches + 1 ¼ inches + ½ inch* for a person who knows the decimal equivalents of the fractions and a person who doesn't.

Person #1

2.125 + 1.25 + .5 = 3.875 (which is 3 ⅞ inches)

Person #2

2 M+ 1 ÷ 8 = M+
1 M+ 1 ÷ 4 = M+
1 ÷ 2 = M+
MR

(Now the person has the difficulty of determining what fraction of an inch the display corresponds to.)

Mathematical Principles

Mathematical concepts and generalizations will continue to be of great importance in the age of technology. People still need to know things like the different meanings of subtraction and the commutative property, not only for their general understanding of mathematics, but for practical reasons as well. The sequence of keystrokes becomes more meaningful for Person #2 in the example above, for instance, if he knows that 2 ⅛ (2 + ⅛) is the same as ⅛ + 2. (The sequence 1 ÷ 8 + 2 = M+ puts the intermediate result—2.125—on the display, while the strict left to right sequence hides it in memory, increasing the chance of error.)

Decimals

Children are now being exposed to decimals at an earlier age than ever before. For example, if they use the calculator to find out how many candies each of 5 children will get if a bag of 27 candies is divided equally among them or if they want to make the turtle in a Logo program turn 7 ½ degrees, they will need some knowledge of decimals.

Since many everyday mathematical situations that confront children cannot easily be expressed as decimals, it is recommended that children be first introduced to rational numbers as fractions in the first and second grades. They should read, write, and understand numerals like ⅓ and 2 ⅛. After children understand these numbers and base ten numeration (through the thousands), decimal numeration for numbers in tenths and hundredths should be taught. Finally, children should memorize the decimal equivalents of common fractions (and vice versa).

Pencil and Paper Computations

An important use for pencil and paper in the future will be to help children learn a variety of mental and calculator computational procedures. For example, we may wish to teach children to add multidigit numbers from left to right using pencil and paper rather than from right to left as has been recent practice because the former algorithm leads directly to the best strategies for mental addition. (See Chapters 6 and 10.) These pencil and paper strategies will also come in handy if no calculators or computers are available and they need to do some complex calculating.

The pencil and paper computations we teach should be of the type that are meaningful (i.e., do not have many obscure steps which in themselves have little meaning like our long-division algorithm), and are easy to remember so we do not have to waste much time teaching them. They should be appropriate for computations with small numbers, but also expandable to longer computations in a pinch. An example is the "subtractive algorithm" discussed in Chapter 11.

Estimation and Mental Computation

Aside from the basic facts, little time is currently spent teaching mental computation. This area of the curriculum should receive much more emphasis since it is important to be able to determine if the answers obtained with computing devices are accurate. Mental computation is also very important when approximate results are needed or when the numbers being operated on are small or powers of 10, and so forth. In the recent past, most people have had to develop their own mental computation techniques. The result of the neglect to teach the most efficient techniques is that many people who have not developed successful mental computation techniques become frustrated in daily tasks as simple as making change. The failure to provide people with reliable mental computation techniques, I believe, is largely responsible for the math phobia suffered by many people. Chapter 6 and later chapters in the book present suggestions for developing these abilities in our students.

WHAT KIND OF CALCULATOR TO GET?

The best calculator for the elementary classroom is one costing between 3 and 10 dollars—the kind they sell in most department stores. It should have fairly large and well-spaced keys and a liquid crystal display (LCD)—these use less energy than light-emitting diode (LED) displays. Data entry should be *algebraic* (i.e., problems are entered from left to right as the values appear in algebraic sentences—to add 35 and 20, you enter 35 + 20 =) rather than *reverse Polish* (i.e., the values in the problem are entered first, followed by the operation—to add 35 and 20, you type 35 <ENTER> 20 <ENTER> +). Almost all inexpensive calculators use algebraic entry.

The best source of power is the sun—solar-powered calculators never need their batteries replaced or recharged. The next best energy source is mercury batteries that last about a year. Battery-operated calculators that turn themselves off if students forget can save your school a lot of money.

The keys you will need on your calculator are the four basic operations (+, −, ×, ÷), M+, M−, MR (add to memory, subtract from memory, and recall memory), and C (clear). A square root button is very useful for upper elementary students or if you are doing any statistical computations. A percent key is helpful at income tax time, though you will need to read your instruction booklet very carefully. Don't assume that when you figure out how to use the percent key on your calculator that other brands work the same for operations like calculating a 25% discount—they do not.

For group instruction in the classroom, calculators that include the features mentioned above are available for use on an overhead projector. Check teacher supply stores or mail order companies for these products.

The calculator is one of the best inventions ever for the arithmetic teacher—it frees you to teach mathematics! Make the calculator an integral part of your classroom and remember its pervasiveness in the real world when you decide what types of mathematical skills to teach.

SUGGESTED READINGS

Arithmetic Teacher 23 (7), March 1976. Contains 11 articles dealing with calculators.

Arithmetic Teacher 34 (6), February 1987. Focuses on calculators and mathematics instruction.

Bartalo, D. B. Calculators and problem-instruction: They were made for each other. *Arithmetic Teacher* 30 (5), January 1983: 18–21.

Bitter, G. G., and Merkseel, J. L. *Activities Handbook for Teaching with the Hand-held Calculator.* Boston: Allyn and Bacon, 1980.

Carpenter, T. P., et al. Calculators in testing situations: Results and implications from national assessment. *Arithmetic Teacher* 28 (5), January 1981: 34–37.

Duea, J., and Ockenga, E. Classroom problem solving with calculators. *Arithmetic Teacher* 29 (6), February 1982: 50–51.

Lappan, G., Phillips, E., and Winter, M. J. Powers and patterns: Problem solving with calculators. *Arithmetic Teacher* 30 (2), October 1982: 42–44.

Mayer, P. I. When you use a calculator you have to think! *Arithmetic Teacher* 27 (5), January 1980: 18–21.

Morris, J. P. Problem solving with calculators. *Arithmetic Teacher* 25 (7), March 1978: 24–25.

Ockenga, E. Chalk up some calculator activities for rational numbers. *Arithmetic Teacher* 31 (6), February 1984: 51.

Suydam, M. N. Achieving with calculators. *Arithmetic Teacher* 31 (3), November 1983: 20.

Wheatly, G. Calculators in the classroom: A proposal for curricular changes. *Arithmetic Teacher* 28 (4), December 1980: 37–39.

Wiebe, J. H. Calculators and the Mathematics Curriculum. *Arithmetic Teacher* 34 (6), February 1987: 57–60.

Wiebe, J. H. Using a calculator to develop mathematical understanding. *Arithmetic Teacher* 29 (3), November 1981: 36–38.

QUESTIONS

1. Discuss how the calculator could be used to help children develop the ability to count forwards and backwards by 2s, 3s, or 5s, etc.
2. What are the advantages and disadvantages of using more powerful calculators (those with scientific functions or several memories, etc.) in the classroom?
3. Go to a department store and compare several calculators in the $3 to $6 range. Which would you choose for your classroom? Why?
4. Find an article from the morning newspaper that involves mathematics. Describe a lesson using the newspaper clipping and calculators.
5. Find the square root of 71 to the nearest thousandth using the estimation and trial method discussed in this chapter.
6. Use your calculator, the newspaper, and an encyclopedia to solve the following problem: It has been stated that all the gold ever mined in the world would make a cube 10 yards on a side. How much would that gold be worth using today's price (New York market)?
7. It has been argued that we should not teach fractions at all because of the metric system and technology. Defend or refute this position.

CHAPTER

4

Computers in Mathematics Instruction

Ever since the invention of the computer, people have been predicting that it would revolutionize education. But if you walk into almost any school in the country, it appears that the same types of things are happening that were happening 30 years ago: teachers are presenting information to children, children are doing pencil-and-paper seatwork, or they are doing small-group activities. Looks are deceiving: subtle changes are occurring in our curriculum and methodologies because of this machine, whether or not there are many in the classroom. Moreover, the trickle of computers going into schools is becoming a flood. Soon, most teachers will have computers available for use in their teaching. Even those who don't have computers available in their own classrooms will see changes in mathematic content and methodology because of the computer.

In this chapter, we first discuss the impact that computers will and should have on mathematics content and on the way mathematics is taught. In addition, we suggest uses for computers to enhance mathematics instruction, particularly in (1) classroom management, (2) the development of understanding of mathematics, and (3) computer-assisted instruction in the elementary mathematics classroom.

COMPUTERS AND THE MATHEMATICS CURRICULUM

Several factors will influence the impact that computers will have on mathematics curriculum. One of these is the importance that school boards, state boards of education, and textbook publishers give to computer literacy, and the way they define it. *Computer literacy,* as defined by various groups, may include any or all of the following: knowledge of what computers are and how they work; knowledge of computer-related terminology; knowledge of how computers are used in our society; ability to use prewritten software to carry out relevant tasks; or, ability to program computers to carry out given tasks.

An important debate taking place is whether or not children should be taught programming skills and concepts. Certain educators argue that few adults will ever need to be able to program in BASIC, Logo, or Pascal. As computers become more sophisticated and powerful, they will be able, for the most part, to write their own programs from natural-language instructions (English, Spanish, etc.) given them. They also argue that it is more important to teach students what computers are and how to use software like word-processing programs or graphics packages to accomplish personal or business-related tasks. Others, like Seymour Papert (1980), believe that programming/problem solving on the computer should be a part of every child's school experience. They claim that the importance of programming does not relate to the development of skills for the marketplace but rather to learning to think, solving problems, and exploring mathematics and other topics. In this view, the computer becomes more than a smart typewriter or filing cabinet—it becomes a tool for the mind.

The viewpoint of this book is that the programming/problem-solving activities can and should be a part of every child's experience with computers and mathematics. Logo and other languages (e.g., BASIC and simplified versions of Pascal like "Karel the Robot") provide powerful tools with which to explore many mathematical concepts. In addition, while children are programming, they are learning problem-solving skills, they are learning to use the computer as a tool to accomplish tasks of interest to them, and they are learning the capabilities and limitations of computers.

Regardless of the way computers are eventually used in our schools, they (along with calculators) will have a major impact on when we introduce decimals. Virtually no computer software packages or computer programming languages use fractions to represent rational numbers, nor will they in the future. The only place where numerals like "$^3/_4$" are used in conjunction with computers are programs designed to teach fractions to children. Thus, a third grader learning about division might write the following statement to find the answer to a division word problem:

```
PRINT 15/4
```

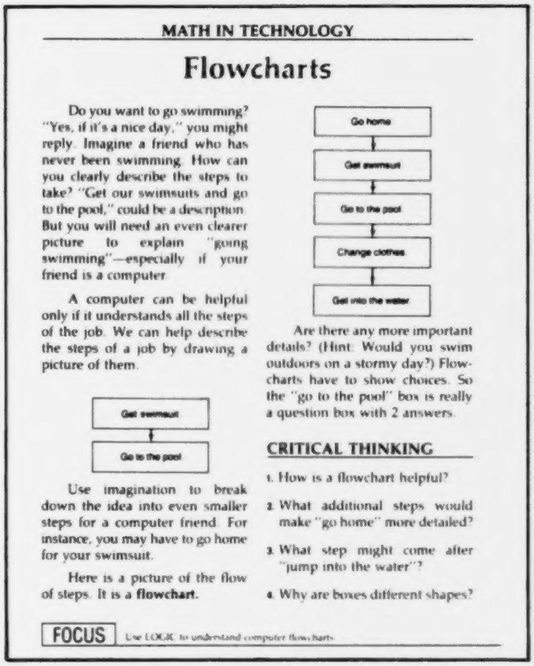
This student should be prepared to understand that the output for the problem, namely, 3.75 is the same as $3^3/_4$. Even if the student never writes a single computer programming line in her life, she needs to understand decimals in order to use much of the prewritten software like VISICALC, programs designed to teach students about science, social studies, simulations, and so on.

We, need, therefore, to introduce decimal numeration to children at an earlier age than we now do. Instead of attempting to make sure that children can do operations on fractions before introducing them to decimal numeration—which may delay introduction to decimals until junior high for some students—we should introduce them after students have mastered whole number numeration, the concept of fraction (part-whole meaning), and fractional numerals like $^1/_2$, $^3/_4$, $^3/_8$, and $^5/_4$. Thus, if children are introduced to beginning fraction concepts and whole number numeration in the first grade, and if these concepts are mastered at the beginning of the second grade, the logical time to introduce decimal numeration is in the middle of the second grade. Chapter 12 of this book gives a detailed plan for introducing decimal numerals to young children.

If many school districts around the country begin asking that elementary schools introduce children to computers, including discussions of what computers are and how they are used, and if they include computer-related activities like flowcharting, writing noncomputer "programs" that do things like moving "robots" through mazes, and using computer vocabulary, then other changes in mathematics curriculum will happen. It is interesting to note that many of the newest editions of K-8 mathematics textbook series include the following elements.

Flowcharting

Not only do flowcharting activities help children understand how computers work and the type of tasks a programmer must carry out, the flowchart serves as an excellent problem solving tool, with or without a computer. The act of flowcharting a problem requires careful analysis of the problem situation and detailed listing of the steps needed to find a solution—strategies that have been shown to improve problem solving performance in students.


Go home
Get swimsuit
Go to the pool
Change clothes
Get into the water
Get swimsuit
Go to the pool

MATH IN TECHNOLOGY

Flowcharts

Do you want to go swimming? "Yes, if it's a nice day," you might reply. Imagine a friend who has never been swimming. How can you clearly describe the steps to take? "Get our swimsuits and go to the pool," could be a description. But you will need an even clearer picture to explain "going swimming"—especially if your friend is a computer.

A computer can be helpful only if it understands all the steps of the job. We can help describe the steps of a job by drawing a picture of them.

Use imagination to break down the idea into even smaller steps for a computer friend. For instance, you may have to go home for your swimsuit.

Here is a picture of the flow of steps. It is a **flowchart**.

Are there any more important details? (Hint: Would you swim outdoors on a stormy day?) Flowcharts have to show choices. So the "go to the pool" box is really a question box with 2 answers.

CRITICAL THINKING

1. How is a flowchart helpful?
2. What additional steps would make "go home" more detailed?
3. What step might come after "jump into the water"?
4. Why are boxes different shapes?

FOCUS Use LOGIC to understand computer flowcharts.

Used with permission of Scribner-Laidlaw Educational Publishers, a division of Macmillan Inc. From Laidlaw Mathematics, Series 2000, Grade 3, by Clarkson et al. Copyright 1987 Laidlaw Brothers Publishers.

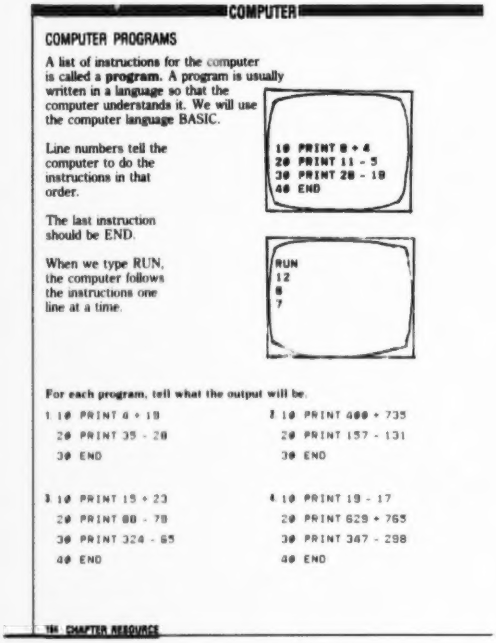
Base Two and Other Bases

The introduction to other base number systems, namely base two and base eight, will help children understand how data is stored in computers. It can also give them valuable insight into our base-ten numeration system.

Simulated Programming

Some textbooks now include activities wherein students write simple programs to send a robot through a maze to reach a desired goal. This type of activity helps prepare children to learn Logo, BASIC, or other computer languages. Others include programming activities in BASIC or LOGO that help students begin to use logical, sequenced strategies for solving problems.

All over the country children and their parents are asking to learn how to program the personal computers they have at home. They want the power to make the machines do exactly as they want. In addition, many educators believe that computer programming activities promote general problem-solving capabilities. Finally, programming activities can be used to teach many mathematical concepts.



COMPUTER

COMPUTER PROGRAMS

A list of instructions for the computer is called a **program**. A program is usually written in a language so that the computer understands it. We will use the computer language BASIC.

Line numbers tell the computer to do the instructions in that order.

```
10 PRINT 8 + 4
20 PRINT 11 - 5
30 PRINT 28 - 19
40 END
```

The last instruction should be END.

When we type RUN, the computer follows the instructions one line at a time.

```
RUN
12
6
7
```

For each program, tell what the output will be.

```
1. 10 PRINT 6 + 19
   20 PRINT 35 - 28
   30 END

2. 10 PRINT 400 + 735
   20 PRINT 157 - 131
   30 END

3. 10 PRINT 15 + 23
   20 PRINT 88 - 78
   30 PRINT 324 - 65
   40 END

4. 10 PRINT 19 - 17
   20 PRINT 629 + 765
   30 PRINT 347 - 298
   40 END
```

184 CHAPTER RESOURCE

From the Siver Burdett & Ginn Mathematics Series, Grade 4. Copyright 1987 by Siver Burdett & Ginn, Morristown, NJ. Used with permission.

Because of this, our mathematics curriculum is changing to include mathematics that will help children understand programming concepts and computer input and output. The following are some of the new mathematical concepts that need to be included if students will be programming computers in Logo, BASIC, or other general-purpose computer languages.

Variables

The concept of variable will be introduced at an earlier level than it is now. Using "open number sentences" is a valid way of accomplishing this.

$\triangle = 25$

$*** = \triangle - 8$

What is ***?

Order of Operations

In order to program the computer to obtain the kind of results we want, we need to know the order in which the computer executes mathematical operations. What will the output be in the following Logo command?

```
PRINT 3 + 5 * 2 ^ 5
```

(*Note:* the "*" means multiply in Logo and most other computer languages.)

Use of Parentheses in Mathematical Expressions

When studying hierarchies of operations we should include the use of parentheses for changing the order of operations. How do you get the computer to add 3 and 5 in the above Logo command before anything else?

Angles and Geometric Shapes

When children solve graphics problems using turtle graphics in Logo, they develop understanding of angles and degrees. Other geometric concepts such as "square" and "triangle" should first be experienced through concrete activities before beginning more abstract, computer activities with them. In general, many geometric concepts, such as angles and regular polygons, will be introduced at an earlier age than they are at present as Logo problem solving becomes common in the elementary school.

Coordinate Graphics

In order for students to engage in BASIC graphic programming and to use the SET command in Logo, they need to have some understanding of graphing and coordinate systems. Research by the author shows that average third grade students have little difficulty mastering a coordinate system like the one in Applesoft low resolution graphics.

Boolean Algebra

Boolean algebra and truth tables will need to be introduced to more advanced students to help them understand IF and DO WHILE statements.

Subscripted Variables and Arrays

For more advanced students in the upper grades, subscripted variables are preparation for use of arrays in programming.

Many of the above changes are already occurring in our textbooks, but it is up to teachers to incorporate these changes into classroom instruction. Do not make the mistake of insisting that students master the "facts" and computational strategies before going to what to them are the more interesting and relevant topics listed above. Mathematical principles, problem solving, and new technology-related topics should be studied together with more traditional topics and skills. Indeed, some of the newer topics may motivate more serious study of the "facts" and the standard algorithms for students who see their value in application to computers.

COMPUTERS AND MATHEMATICS TEACHING METHODS

Twenty years ago when computers were first making an impact, some futurists said that computers would replace teachers—a large change in methodology, indeed. They predicted that through the computer, every person would receive personalized instruction at his level of knowledge and at his own pace, geared toward his learning style, and using the type of reward system that results in the greatest amount of learning. Twenty years later, we are still waiting for this momentous change.

It will be a long wait. Computers will not replace teachers in this century, and probably not in our lifetimes. In most school districts, a typical teacher might expect to have a single computer in his classroom two or three years from now, and if he is lucky, a computer resource center where he can send students for enrichment, practice, or diagnosis. Computers will supplement teacher instruction. They will be used to do things that the teacher has difficulty doing in a regular classroom setting, such as individually teaching a child who has missed a week of class because of illness, diagnosing the problems of a child needing remediation, and motivating a reluctant learner. Software is already available to help meet these needs. However, neither the software nor hardware is presently available for the masses that will provide anything close to a complete education.

COMPUTER MANAGEMENT OF YOUR CLASSROOM

The computer has the potential to greatly ease the amount of busywork for any professional. For the teacher, the computer can save time by generating and grading tests, recording and retrieving student data, grouping students, and organizing her day.

Many major publishers now sell programs for generating worksheets and examinations. You need a printer and probably a disk drive to use your computer this way. Usually, these programs start with a "menu" where you select the types of problems you wish to include on the test or worksheet, and the program randomly generates a tests for you, perhaps with different items on each test and with accompanying answer sheets. Beware, however; such programs usually do not include tests of concepts since such items usually require pictures, manipulatives, and/or verbal explanations and questions. Thus, they tend to cover the lower levels of learning—recall of algorithms and "facts."

Another classroom management tool is the spreadsheet. Computer spreadsheets were developed for use by accountants for the purpose of keeping financial records and for forecasting the financial results of possible changes in production, pricing, salaries, and so forth. They have proven very useful in other areas as well.

Before the computer, a spreadsheet was simply a large sheet of paper containing rows and columns of financial information. AppleWorks, Lotus 1-2-3, VISICALC and other comparable products put the spreadsheet in the computer and gave it more power. Think of the computer spreadsheet as a large sheet with columns and rows that can be filled with numerical information about students. The computer screen provides a "window" that allows you to scan the spreadsheet looking at a few rows and columns at a time. The program allows you to input or change data, give headings to columns or label rows, input formulas for totaling columns and rows, or more complex computations like finding averages or assigning grades.

Suppose, for example, that you wanted to keep track of your students' math grades on a spreadsheet. You might put their names in the first column, scores for homework and tests in the next 15 columns, formulas for computing averages in the 17th column, and formulas for computing a letter grade in the 18th column. In addition, you might want to put formulas for calculating the average grade at the bottom of test columns. Finally, you might want to insert a title for the spreadsheet and column headings at the top. Figure 4.1 contains a spreadsheet with these attributes created with AppleWorks, Apple Corporation's integrated applications package for the Apple II line of computers.

FIGURE 4.1
A spreadsheet used for recording grades.

File: EMTA1

Page 1
7/12/87

Last Name	First Name	Asg 1	Asg 2	Asg 3	Asg 4	Asg 5	Test 1	Asg 6	Asg 7	Asg 8	Test 2	Asg 9	Asg 10	Test 3	TTL Pts	Avg.
Adams	Jane	9	8	9	9	10	85	6	8	10	88	6	9	94	351	87.750%
Avery	Alvin	6	8	7	9	10	85	8	9	10	90	5	10	87	347	86.750%
Boudreaux	Bernice	8	7	7	8	8	72	8	6	10	80	8	9	95	326	81.500%
Cedera	Juanita		7	6	7	7	78	8	9	9	80	9	9	82	311	77.750%
Chung	Connie	4	6	4	8	6	62	7	8	6	78	10	8	77	284	71.000%
Connand	Barbara	10	10	10	9	10	99	10	10	10	97	8	10	100	393	98.250%
Dovery	Jason	8	9	10	10	7	90	8	9	10	92	10	9	94	366	91.500%
Gradey	Charlotte	10	8	8	9	10	88	7	10	9	89	10	8	90	356	89.000%
Halley	Peter	7	7	5		9	77	6	8	8	75	8	10	79	299	74.750%
Johnson	Jennifer		5		9	10	55	7	5		67	7	8	70	243	60.750%
Kemp	George	9	9	9	9	10	92	8	9	9	95	8	9	96	372	93.000%
Landry	Steven	8	10	10	8	9	91	9		10	92	9	10	93	359	89.750%
Luvalle	Marcia	10	10	10	10	10	98	9	10	10	88	10	10	94	379	94.750%
Martinez	Raul	5	8	6	8	9	88	8	9	10	89	7	7	89	343	85.750%
Montinez	Jorge	9	9	9	9	9	97	10	9	9	96	8	9	92	375	93.750%
Oleary	Semantha	8	9	9	9	9	91	8	7	4	82	6	3	67	312	78.000%
Pieau	Mensu	5		7	9	10	88	7	5	10	78	9	9	83	320	80.000%
Quiring	Kalvin	8	9	10	10	10	99	10	7	10	93	9	8	97	380	95.000%
Rousseau	Albert	10	10	10	10	10	99	10	10	10	97	10	10	100	396	99.000%
Vasquez	Jeanne	8	7	10		8	79	9	10	8	82	8	7	85	321	80.250%
Yee	Faith	9	10	9	9	10	93	9	9	10	95	9	10	97	379	94.750%

Note that in the *cells* (individual spreadsheet locations) in the last column, formulas have been entered to calculate averages and grades. The formulas themselves do not show inside the spreadsheet on the screen or on a printout: rather, the computed values appear. Most spreadsheets show formulas entered into cells at the bottom of the screen when the cursor (the highlighted area that indicates which cell you are dealing with) is moved to that cell. The formula that has been entered into Cell P5 (cells are identified by a letter to indicate the column and a number to indicate the row) is "@SUM (C5...05)." This formula tells the computer to find the sum of the values in row 5 starting at column C and ending at column O. Note that if you want to weight any of the columns differently from the others, instead of using the built-in average function, you could use all the cell names in your formula and perform whatever operation you want on them. An example of that kind of formula is "(3*B1 + 2*B3 + B*5) /6."

The command used to calculate students' grades is the *IF* statement similar to those used in computer programming languages. Like most computer spreadsheets, AppleWorks is not capable of printing words or letters based on the outcome of *IF* statements. It can, however, print numbers is such a situation. For this reason, the numerical grading system—4, 3, 2, 1, 0—is used instead of a letter grading scheme. Appendix B contains a tutorial for using AppleWork's spreadsheet program.

Another computerized classroom management tool is the data-base program. A data-base program is to words what a spreadsheet is to numbers. It was designed to computerize the file cabinet. Data bases are much more powerful than file cabinets, however, because they allow the user to search and sort on a number of different keys (file cabinets are arranged in one and only one order—usually alphabetical). They often do arithmetic as well. They are a godsend for those of us who must keep data on a large number of people—employees, acquaintances, or students. They allow you to enter names, addresses, and a large amount of personal data, if the information is set up so that whatever part of it you need can be dumped to the printer. This allows you to print complete mailing lists, or to group lists of names. For example, if you had keyed in a variety of information on your class including CTBS reading scores (and if you had the right kind of filing program), you would be able to generate a list of all children below a certain score with a few keystrokes. Filing programs will allow you also to keep track of the materials in your classroom and to sort them or organize them as needed.

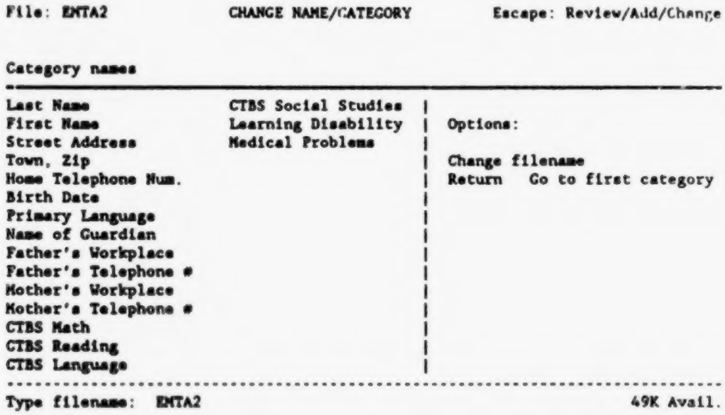
If you wanted to set up a data base on your students, the first step would be to decide what categories of information you would like to keep about each individual. You would also decide the kinds of codes you want to store in the categories. For example, if one of the categories is "Math Group" and you have three groups in your class—the Muppets, the Care Bears, and the Smurfs—you need to decide whether you want to store only the first letter of each group name, the entire name or another code. The reason for this is that when you decide to select and print all students who are in a particular group, you need to enter the code for that group. If you have used several different codes to refer to a group, you have to type each code used. Worse yet, if you forget one or more of the codes or spellings you used, they may be left off your list.

Once you decide upon the categories of data that you want to store in your data base, you set up the format of the file. You do this by typing the names of the categories. Some data bases also allow you to specify the format of the information you enter, for example, that the numbers in a particular category can have 4 digits to the left of the decimal and 2 to the right or that words are limited to 15 letters. Figure 4.2 shows the AppleWorks set-up screen for a fictitious fourth grade class. Figure 4.3 shows the information entered for one individual's records in this math class. Appendix C contains a tutorial for using AppleWork's data-base program.

Most people find that the most useful of all computer applications is the word processor. This type of program turns your computer into a smart typewriter, allowing you to enter text, then store, retrieve, and manipulate it. For example, suppose you wish to create a test or an assignment sheet. You might enter it into the computer, make a printout, read it and discover a variety of needed changes, make the changes, and then print a final draft. Most word processors allow you to insert or delete text, move text around in the document, search for sequences of characters, and tell your printer to underline text or to print it in italics. In addition, many word processors may be combined with spelling or grammar-checking programs to produce a better final product. If you create handouts, tests or other documents that you use over and over with minor modifications each time, you will find the word processor to be an extremely useful tool.

Graphics and fancy text programs are also useful management tools. Fancy text programs allow you to create professional looking overhead transparencies, worksheets, signs, bulletin boards, announcements, letterheads, fliers, and so on.

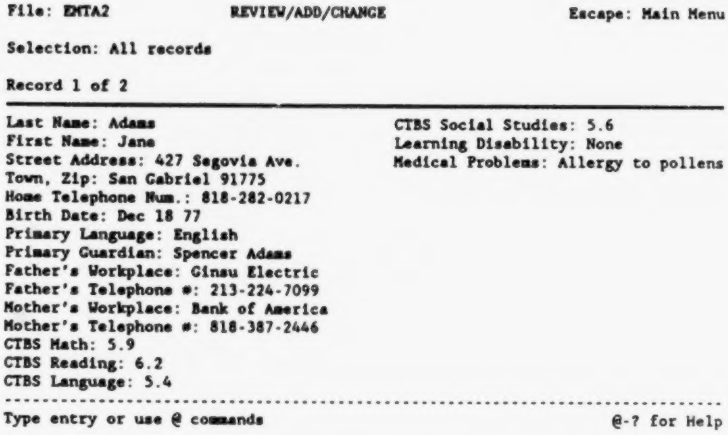
FIGURE 4.2
Data base categories for student data.



```
File: EMTA2              CHANGE NAME/CATEGORY          Escape: Review/Add/Change

Category names
--------------------------------------------------------------------------------
Last Name                CTBS Social Studies  |
First Name               Learning Disability  |  Options:
Street Address           Medical Problems     |
Town, Zip                                     |  Change filename
Home Telephone Num.                           |  Return    Go to first category
Birth Date                                    |
Primary Language                              |
Name of Guardian                              |
Father's Workplace                            |
Father's Telephone #                          |
Mother's Workplace                            |
Mother's Telephone #                          |
CTBS Math                                     |
CTBS Reading                                  |
CTBS Language                                 |
--------------------------------------------------------------------------------
Type filename:  EMTA2                                                49K Avail.
```

FIGURE 4.3
A data base record for one student.



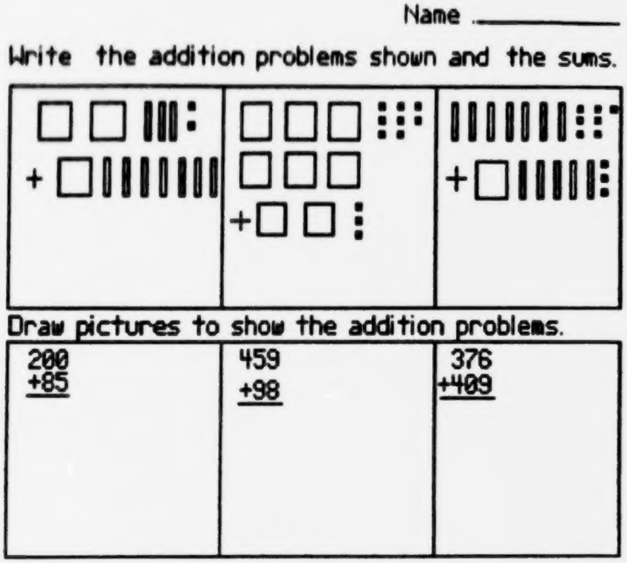
```
File: EMTA2              REVIEW/ADD/CHANGE                    Escape: Main Menu

Selection: All records

Record 1 of 2
--------------------------------------------------------------------------------
Last Name: Adams                             CTBS Social Studies: 5.6
First Name: Jane                             Learning Disability: None
Street Address: 427 Segovia Ave.             Medical Problems: Allergy to pollens
Town, Zip: San Gabriel 91775
Home Telephone Num.: 818-282-0217
Birth Date: Dec 18 77
Primary Language: English
Primary Guardian: Spencer Adams
Father's Workplace: Ginsu Electric
Father's Telephone #: 213-224-7099
Mother's Workplace: Bank of America
Mother's Telephone #: 818-387-2446
CTBS Math: 5.9
CTBS Reading: 6.2
CTBS Language: 5.4
--------------------------------------------------------------------------------
Type entry or use @ commands                                     @-? for Help
```

Graphics programs allow nonartists and artists alike to use the computer to create pictures, designs, drawings, and worksheets. Most also have the capability of creating text using different fonts and text sizes. Many graphics/fancy text programs have a built-in set of graphics characters that can be called up and placed on the page as desired. Figure 4.4 shows a math worksheet created with the Mouse Paint.

Another major category of general applications software that is useful in classroom management is the *communications package*. These, along with a modem, allow you to use your computer to communicate with other computers. A *modem* translates the electronic signals in your computer to audio signals that may be transmitted across telephone lines (Figure 4.5). You can have access to numerous data bases, bulletin boards, and other information sources on computers that may be useful for a variety of purposes in the classroom. For example, if you want to find out what research has been done about teaching problem solving to first graders, you could use your communications package to tap into the ERIC data



Name ______________

Write the addition problems shown and the sums.

Draw pictures to show the addition problems.

200 +85	459 +98	376 +409

FIGURE 4.4
A worksheet created with Mouse Paint.

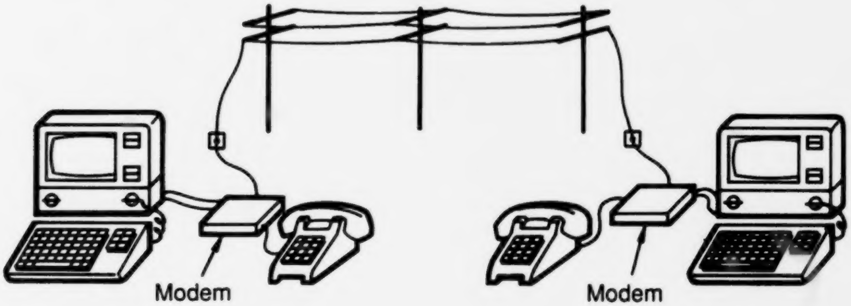


FIGURE 4.5
Computer communications.

base. One of the hottest new uses of the computer is for teachers to exchange information using computer communications. Many districts are establishing computer bulletin boards so that teachers can access information about school events, tap into resource data base, or send electronic messages. For example, fifth and sixth grade mathematics teachers in a district might enter teaching suggestions, lesson plans, worksheets, or games into the bulletin board so that they could be used by others. With proper equipment and software, teachers would be able to download and print worksheets, graphics, lessons, and so forth, from the bulletin board in their classroom.

A variety of programs now available combine several of the above applications. AppleWorks, for example, contains a word processor, a data base program, and a spread sheet. Lotus 1-2-3, a program for the IBM PC and similar computers, contains a spreadsheet, a data base, and a graphics program. This type of program allows you to create information in one application and use it in another. For example, you could transfer detailed information about children in your class (previously stored into the data base) to the word processor and then generate letters to their parents about the progress of their children in your class.

TEACHING MATHEMATICS WITH COMPUTERS

Using Programming Activities in Teaching Mathematics

When children learn a computer language like Logo, they possess a powerful tool for exploring and reinforcing many mathematical relationships and principles. The level of programming needed to explore mathematical problems and relationships is not high. The concepts presented in Chapter 7 of this book can be covered in approximately 15 forty-minute sessions for elementary students (approximately 5 sessions for adults). The basic tools the students need are a knowledge of the PRINT, REPEAT, IF, READLIST commands, and of the graphics commands in Logo (or equivalent commands in other languages such as BASIC) and an understanding of the use of variables. Programming activities in Logo for developing mathematical understanding are integrated throughout the rest of this book. We strongly suggest that you do the activities in Chapter 7 or Appendix A (the equivalent BASIC introduction) before proceeding to the methods sections of this book.

An example of the use of Logo in developing understanding of a mathematical operation is to ask children to use the *REPEAT* command and repeated addition to find the product of two numbers. If the problem were *128* × *107*, the resulting procedure might look like the following:

```
TO MULTIPLY
   MAKE "FACT1 128
   MAKE "FACT2 107
   MAKE "PROD 0
   REPEAT :FACT2 [MAKE "PROD :PROD + :FACT1]
   PRINT :PROD
END
```

Note that this activity could be extended by asking your pupils to write the procedure so that it would use repeated addition to find the product of any two whole-number factors.

The most frequent use of Logo in the mathematics classroom is probably in teaching geometric concepts. Children are highly motivated by activities where they write Logo commands to move the "turtle" around the screen to create shapes and scenes. If properly sequenced and developed, these activities can lead to the discovery of many geometric concepts. An important feature of Logo Turtle Graphics is that Turtle Graphics are "relative." That is, in the graphics mode usually introduced to young children, the movement of the "turtle" (actually, a triangle that appears on the screen) around the screen is relative to its current location and heading. For example, if we start the turtle in the "home" position (the center of the screen) and ask it to move forward 50 steps, then turn right 30 degrees as shown in Figure 4.6, the next turtle movements would be relative to its current position.

This type of graphics lends itself well to modeling. A common activity is to have children move around a "screen" drawn on the floor to show what would happen on the screen when graphics commands are executed. The children then imagine their movements to help them understand and plan the movement of the screen "turtle."

Using Applications Programs in Teaching Mathematics

Word processors, spreadsheets, and other general applications programs are not limited to use in classroom management. They make outstanding problem-solving tools for students as well. Many computerized applications programs, such as Bank Street Writer or PFS: Plan, are simple enough to be used by children. Moreover, as children use these tools to solve problems, not only are they learning problem-solving skills, but they are learning skills that will be useful to them in later life.

Perhaps the most useful applications tool in mathematics is the spreadsheet. Many problem-solving tasks that require students to operate on rows and columns of data could be handled by a computerized spreadsheet. For example, suppose you asked children to gather data about their classmate's heart rate under different conditions, then to state generalizations about the effect of these conditions on heart rate. After the data had been gathered, the students could enter the data into a spreadsheet, calculate averages, then discuss the results. Some spreadsheets would even generate graphs from the data.

Other computer applications programs can be of use in math instruction as well. A word processor could be used to have children write problems for themselves and other children to solve, to write the steps as they are solving problems, or to write reports on integrated science-mathematics or social studies-mathematics activities. Communications packages could tap into data bases or computerized encyclopedias, for example, to get information for solving applied math problems. Data-base programs could store and manipulate non-numeric data gathered in problem solving or integrated activities. Graphics packages like Micro Illustrator (used with the Koala Pad) or Mouse Paint could be used by children to draw models for mathematical operations. Numerous articles in such journals as *The Computing Teacher* and *Arithmetic Teacher* give suggestions for using applications packages in the classroom.

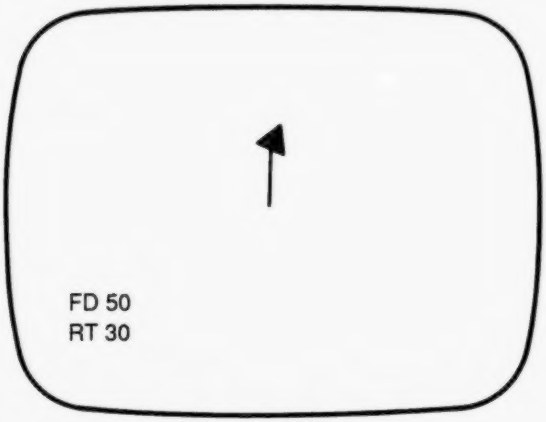
Using the Computer as a Tutor

Software for "teaching" elementary mathematics is everywhere. Almost any home computer store, software store and even department store sells programs dealing with addition, division, or decimals. But beware. Few of these programs do what they really should—make use of the capabilities of the computer to develop understanding of mathematics through interactive instruction, model mathematical concepts through screen graphics, diagnose student responses and branch to appropriate levels of instruction, or orient instruction to the appropriate learning style.

One type of microcomputer software simply uses the computer as an electronic ditto sheet, placing mathematics problems on the screen, accepting the student's answer, and telling the student whether or not the answer is correct. Although some research evidence suggests that computerized drill in mathematics can produce better results than pencil and paper drill, especially for below average students (Forman 1982), it is questionable whether this is the best use of computers in the classroom. Research has also shown that just the novelty of incorporating drill into games can produce better results than traditional drill. In addition, it seems an anachronism to use the computer to drill students on pencil and paper computation when the computer and calculator are reducing the need for this type of skill.

If you do not have unlimited access to computers for your students, you need to decide how the machines can best be used. Should you tie up your $1000 ma-

FIGURE 4.6
A Logo graphics example.



```
FD 50
RT 30
```

chine doing something that can be done equally well with a 2 cent piece of paper or $2.00 game? If you *do* decide to use your computer for drill and practice, rather than spend your or your school's money on drill programs, you may wish to use the drill program listed in Appendix D of this book. It allows you to choose the operation and kinds of numbers included in each exercise appearing on the screen. It tells the student if the answer is right or wrong; for wrong answers, it gives the correct response. All this for free! Rather than using this program to give students drill on pencil and paper computation, however, you are encouraged to use it to give them practice on mental computation.

Of course, there are many better ways of using the computer as a tutor in mathematics. Excellent, commercially-prepared programs are available making use of the capabilities of the computer to give students experiences that would be difficult to duplicate through other means. The different types of instructional programs are as follows:

1. **Tutorials.** These programs teach mathematics to students. The best tutorials teach mathematical concepts and understandings rather than simply the steps in an algorithm. For example, a tutorial in fractions might use graphics to teach the meaning of fractions and the addition of fractions. To do this, it might start with rectangles and other shapes, then, through the use of animation, divide and shade them to represent a variety of fractional numerals. (See Figure 4.7.)

To show addition of fractions, the two fractions being added would be graphically represented using "strips." (See Chapter 14.) They would then be moved together using animation techniques and their sum would be found using number lines. The user could test the sum with any kind of number line she wished by typing in the appropriate command sequence (e.g., number line divided in thirds, sixths, or twelfths). (See Figure 4.8.)

2. **Drill and Practice.** This type of program presents exercises to students, accepts answers, and (usually) tells students whether or not their answer is correct. Drill and practice programs may have a variety of additional features—they may be imbedded in a game, give several chances at typing the correct answers, give hints or a limited amount of instruction if incorrect answers are entered, or save data on students who use the program for later review by the teacher.

3. **Simulations.** These programs take students through a series of situations that simulate real-life experiences. They often incorporate graphics to be more realis-

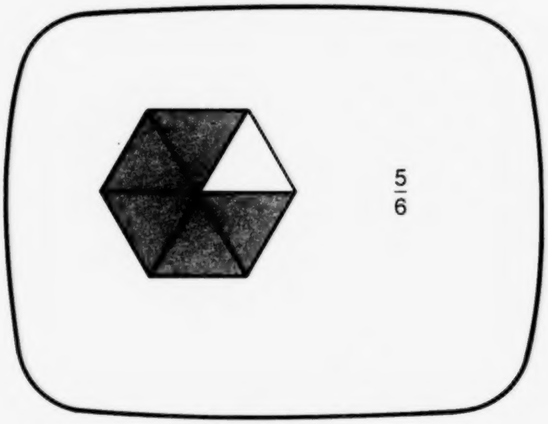
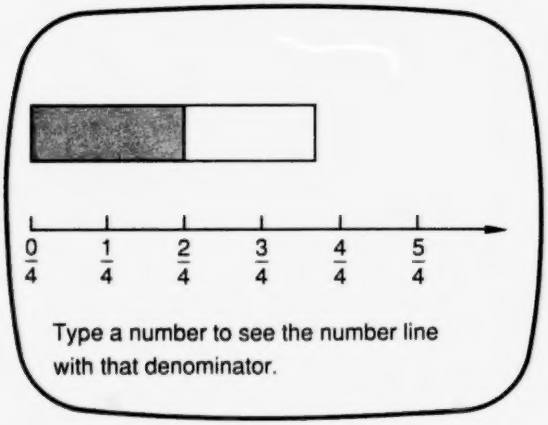


FIGURE 4.7
A graphics tutorial.



Type a number to see the number line with that denominator.

FIGURE 4.8
A graphics model for adding fractions.

tic. Students must solve mathematical problems to advance toward some goal. The direction of the simulated experience changes as the student makes choices and solves problems. For example, a simulated treasure hunt might involve calculating angles, determining distances, and doing operations with decimals in order to find the treasure. A popular simulation involving mathematics is "Lemonade Stand," where children try to make a profit in a simulated business venture.

4. **Problem Solving/Logical Thinking.** This type of program is designed to give students practice in solving mathematical, spatial, or logical problems and to develop students' thinking skills. For example, in "Building Perspective" (Sunburst), students are shown side views of buildings and are asked to visualize what they would look like from the top.

5. **Diagnosis/Testing.** The purpose of this type of program is to gather data on students. Typically, the student sits at the computer and answers questions while the computer saves data about the student that is later compiled into a report for the teacher and/or parent. This type of program has the potential to greatly reduce the testing burden for both the teacher and the student since, based on the student input, it gives students questions at their individual level. Gone will be the days when students had to endure countless problems that were either above or below

their capabilities. Although computerized testing is in its infancy, major test publishers are developing instruments that gather precise data on students skills, knowledge, and weaknesses in a reduced time.

Before buying any piece of instructional software, try it out and compare it with your instructional goals. This may present a problem since many of the available programs for teaching mathematics are available only through mail-order companies. Usually, they will not send you samples of their materials to try out for fear that you will make an illegal copy before returning the disk to them. You are therefore at the mercy of their advertising brochures that may promise "exciting graphics" but instead have monocolor, dull, and non-animated pictures that do not relate to the instruction. Or, programs that claim to "teach understanding of decimals" may simply be drill in the steps of algorithms.

If you are interested in a particular piece of software for teaching mathematics, check to see if your school district has a copy. Another source might be the educational computing laboratory or instructional media center of your local university. Mathematics conferences like those sponsored by the National Council of Teachers of Mathematics usually have numerous displays by publishers of instructional software.

Other good sources of information about instructional software in mathematics are the *Arithmetic Teacher* and *School Science and Mathematics* journals. These and other general journals for teachers offer software reviews by experts in mathematics instruction. People doing these reviews have taken the time to thoroughly test the software. Although the reviewer may have a different perspective from you about how mathematics should be taught, at least these reviews provide you with an initial screening. In addition, the reviews usually cover recently-released software, not available to you through other means. Even if a piece of software for mathematics instruction has received rave reviews, you should try to review it yourself to see if it meets your needs.

The following is a list of considerations when reviewing tutorial software in mathematics.

1. What is the instructional objective of the program (e.g., addition of two-digit numbers with regrouping)?
2. What is the instructional format of the program—drill, simulation, tutorial, or instructional gaming? Will this format be beneficial to the students with whom you intend to use it?
3. How many students are needed to run this program? The majority of tutorial programs are designed for one student, but some games require two or more players.
4. What is the instructional sequence? Is the program didactic—explaining principles or procedures followed by practice—or are students led to discover relationships, principles, or procedures?
5. How good is the screen format? Is it confusing? Is a limited amount of text displayed on the screen at one time? Are graphics clear and appropriate to the objectives of the program (i.e., do they enhance the presentation or are they superfluous)?
6. What reading level is required to use the program? Is it above the level of the audience?
7. How easy is it to run this program? Can the least able of your students run the program immediately after a minimum of explanation by you? Does the stu-

dent need to know how to use the operating system of the computer or does she simply need to put in the diskette and have the rest explained to her on the screen? Does the user need to refer to external documentation (manuals) to run the program? Manuals tend to be written in computerese, and even if they are easy to understand, they tend to get lost.

8. What type of computer equipment and what peripheral devices are needed to run the program? How much memory and how many disk drives are needed? Is a color monitor required? Do you need a printer? Joy sticks? Light pens? Manipulative materials? Programs that require peripherals that you do not have do you little good unless you are ready to spend extra money and buy them.
9. What is the quality of instruction? Does the program develop understanding of mathematical principles? Is instruction at the student's level and pace? Are graphic models used to develop understanding?

Figure 4.9 contains an evaluation form for tutorial software in mathematics. Use it to help you decide whether to purchase software (assuming, of course, that you have the opportunity to review it before buying it). Also, check to see if your school district or state education department reviews software or has a center where teachers may run and evaluate software.

COMPUTER HARDWARE

As a computer/mathematics educator, one of the most frequent questions I am asked is, "Which computer should I buy?" Unfortunately, the answer to this question is, "It depends." It depends on what you plan to do with the computer and the amount of money you want to spend. If you are going to use your computer(s) primarily for Logo programming activities, your needs will be somewhat different than if you plan to manage a school or learning resource center. In addition, in the fast-changing world of personal computers, almost any recommendation is outdated almost as soon as it appears in print. Regardless, a few rules of thumb may help you make your choice.

1. Perhaps the most important consideration in deciding which computer to buy is the presence of quality software for the applications you plan. The two types of computers with by far the most available software (both applications-type software and instructional software) are the IBM PC and the Apple II. There are "compatible" computers, IBM PC type and Apple II type, that run the same software.
2. Avoid newly-released computers, especially if they do not run either Apple II or IBM-PC software. It usually takes more than two years for a good library of software to be developed for a new computer and for all the flaws to be worked out.
3. For eight-bit computers (those similar to the Apple IIe or IIc), you need 128K of memory for many applications. For IBM-type computers, you will need at least 256K of memory for many applications.
4. If you are only running CAI software or teaching Logo or BASIC, you will probably be able to get by with one disk drive. For most other uses of the computer, you will need two disk drives. Most users have found "cassette drive" data storage to be inadequate.

FIGURE 4.9
Software Evaluation Form

Date ____________ Reviewer's Name ______________________________

Program Name ____

Publisher ____

Order Number ______________ Price ________________________

Subject Area ____

Educational Objective ______________________________________

Grade or Ability Level ______________________________________

Execution Time—Average Use _________________________________

Peripheral Devices Needed __________________________________

Printer ________ Joy Sticks ________ Light Pen ________

Speech Synthesizer ________ Other __________________________

TYPE OF PROGRAM

________ Tutorial

________ Drill and Practice (Game format? ________)

________ Simulation

________ Problem Solving/Logical Thinking

________ Diagnosis, Testing

Indicate your level of agreement or disagreement with the following statements by marking the appropriate letters. The letters have the following meanings:

SA —Strongly Agree
A —Agree
D —Disagree
SD —Strongly Disagree

APPROPRIATENESS OF PROGRAM FOR STATED OBJECTIVE(S)

SA A D SD Program accomplishes its instructional objective better than other available media.

SA A D SD There is a definite need for this program in my classroom.

SA A D SD The program makes wise use of the computers in my class.

Comments:

NUMBER OF STUDENTS FOR WHICH PROGRAM DESIGNED

________ One

________ Two

________ Three - Four

________ Small Group

________ Large Group

SEQUENCE OF INSTRUCTION

________ Discovery

________ Explanation/Discussion followed by questions/drill

________ Other (Specify)

________ No Instruction in Program

(continued)

QUALITY OF SCREEN FORMATTING

SA	A	D	SD	Text displayed on the screen is uncluttered and easy to read.
SA	A	D	SD	Graphics are clear and appropriate to instruction.
SA	A	D	SD	Method for inputting answers is clear.
SA	A	D	SD	Student controls pace of instruction.
SA	A	D	SD	Method for continuing program displayed.

Comments:

READING LEVEL

SA	A	D	SD	Reading level is appropriate.

Comments:

EASE OF USE OF PROGRAM

SA	A	D	SD	Program is self-booting.
SA	A	D	SD	Student does not need to refer to external documentation to use program.
SA	A	D	SD	After initial explanation, student can use program without consulting teacher.
SA	A	D	SD	Program handles inappropriate student input effectively.
SA	A	D	SD	Program provides method for help if needed.

Comments:

QUALITY OF INSTRUCTION

SA	A	D	SD	Valid instructional techniques are used to accomplish instructional objective.
SA	A	D	SD	Content is accurate.
SA	A	D	SD	Instruction is adapted to level of user.
SA	A	D	SD	Program gives positive feedback.
SA	A	D	SD	The program is motivational to students.

Comments:

OVERALL RATING:

Outstanding Excellent Good Fair Poor

5. Most applications of the computer, including programming, require the use of a printer. Make sure that the computer you buy has a printer port. If you already have a computer and decide to buy a printer, make sure that the printer is compatible with your computer.

The computer revolution has arrived in the classroom. It is up to the teacher to become informed about computers so that their power can be tapped in the classroom. Make use of the opportunity provided in this textbook to learn about them so you may use them to their fullest potential.

SOURCES OF SOFTWARE

Spreadsheet Programs

1. Bank Street Calc (Broderbund Software)
2. Visicalc (Personal Software Inc.)
3. AppleWorks (Apple Corporation)
4. PFS: Plan (Software Publishing Corporation)

Data Base Programs

1. Bank Street Filer (Broderbund Software)
2. AppleWorks (Apple Corporation)
3. PFS File (PFS Corporation)
4. Twin (Mosaic Software)

Word Processors

1. Bank Street Writer (Broderbund Software)
2. AppleWorks (Apple Corporation)
3. Kidwriter (Spinnaker)
4. MasterType (Scarborough Systems) (a typing tutorial game)
5. The Milliken Word Processor (Milliken Corp.)

Graphics

1. Mouse Paint (Apple Corporation)
2. Mica Illustrator (Koala Technologies)

Math CAI

1. Creative Publications (P.O. Box 10328, Palo Alto, CA 94303) offers a variety of well-reviewed math software developed by other publishers and by Creative Publications. Their catalog gives a brief description of each program they sell. The following programs have been developed by Creative Publications:
   Fraction Bars Computer Program
   Mathematics Life Skills
2. Milton Bradley (P.O. Box 2888, Springfield, MA 01101) has a variety of programs designed to develop understanding of mathematical concepts:
   Ratio and Proportion
   Percents
   Mixed Numbers
   Division Skills
3. CBS Software (One Fawcett Place, Greenwich, CT 06836) has published the *Success with Math* series including:
   Addition and Subtraction
   Multiplication and Division
   Fractions: Addition and Subtraction
   Fractions: Multiplication and Division
   Decimals: Addition and Subtraction
   Decimals: Multiplication and Division

4. Sunburst Communications (Room BD 393, 39 Washington Ave. Pleasantville, N.Y. 10570) has a number of excellent conceptually-based programs in math:
    - Number Quest
    - Challenge Math
    - Explorer Metros
    - Hot Dog Stand
    - The Whatsit Corporation
    - Survival Math
    - The King's Rule
    - Puzzle Tanks
    - SemCalc
    - Teasers by Tobbs
    - The Pond: Strategies in Problem Solving

SUGGESTED READINGS

Arithmetic Teacher 30(6), February 1983. (Has ten articles about computers in the classroom).

Arithmetic Teacher. Each monthly issue has software reviews and a column called the "Computer Corner."

Beechhold, H. F. Computers in Education: Is 'educational' software the answer? *Electronic Education* 2 (3), November 1982: 44-48 and 90-91.

Clements, D. H. Programming, problem solving, and practice. *Arithmetic Teacher* 31 (4), December 1983: 32-35.

Computers in Mathematics Education (1984 Yearbook). Reston, VA: National Council of Teachers of Mathematics, 1984.

Doyle, W. H. A discovery approach to teaching programming. *Arithmetic Teacher* 31 (4), December 1984: 16-18 and 27-28.

The Computing Teacher. Each issue has numerous articles about computer uses in the classroom.

Electronic Learning. Each issue has numerous articles about computer uses in the classroom.

Markuson, C., Tobias, J., and Lough, T. Logo fever: The computer language every school is catching. *Arithmetic Teacher* 31 (1), September 1983: 48-51.

Morris, J. P. Microcomputers in a sixth-grade classroom. *Arithmetic Teacher* 31 (2), October 1983: 22-24.

Saltinski, R. Graphs and microcomputers: a middle school program. *Arithmetic Teacher* 31 (2), October 1983: 17-20.

Wiebe, J. H. Needed: Good tutorial software in mathematics. *School Science and Mathematics* 83 (4), December 1983: 281-292.

Wilburg, K., and Rader, B. Computers are elementary, *Arithmetic Teacher* 31 (5), January 1984: 18-22.

Williams, R. A. Preschoolers and the computer. *Arithmetic Teacher* 31 (8), April 1984: 39-42.

QUESTIONS

1. Will computers have less or more effect on the mathematics classroom than calculators? Why?
2. Find a school that has recently purchased microcomputers and interview the person or group who decided which type of computer to purchase. How did they arrive at the decision to buy that brand?
3. Use the software evaluation form enclosed in this chapter to evaluate a piece of software designed for the mathematics classroom.
4. Obtain the teacher's manual of the fifth-grade book in a K-8 mathematics textbook series published during the last 2 years. What suggestions about computers does it have for the teacher? What computer-related activities does it have for children?
5. Discuss how word-processing programs could be used in the mathematics classroom.

CHAPTER 5

Teaching Problem Solving in Mathematics

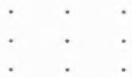
DEFINING "PROBLEM SOLVING"

Most educators agree that problem solving is an important part of the curriculum. The question is, what is *problem solving*? The following are six different mathematical situations that might be classified as "problems."

PROBLEM 1: Susie has 4 books. John has 5 books. How many books do Susie and John have together?

PROBLEM 2: Frank is Susie's brother. John has two sisters, one of whom is Jennifer who is the next oldest child in her family. John is the oldest child in his family. Jennifer and Susie are twins. Who is the youngest child in this family?

PROBLEM 3: Draw four straight line segments without lifting your pencil to hit all of the dots in the array below.



PROBLEM 4: Suppose someone gave you one cent on the first day of May, 2 on the second day, 4 on the third day, 8 on the fourth day, and so on throughout the month. How much money would you receive in May?

PROBLEM 5: How much does it cost to go from Los Angeles to New York?

PROBLEM 6: Round each number below to the nearest tenth.

a. .34
b. .792
c. 4.08

These problems are obviously very different from one another. Problem 1 is typical of the problem-solving exercises that appear in most elementary mathematics textbook series. Some people might not consider 6 to be a problem-solving task because it is not *verbally stated* and does not appear in a *real* context. Some mathematics teachers might feel uncomfortable giving 2 or 3 to a child to solve because they do not cover mathematical operations and there is not a quick, apparent route to a solution. Others might blanch at 5 because it is very open ended—there are a variety of possible answers and routes to obtaining answers.

Actually, any of the above situations can involve problem solving, depending on the background and involvement of the person attempting them. By definition, problem solving occurs when a person attempts to solve "a problem." But, what is a problem?

> ***A problem is a situation that requires resolution but where there is no immediate, clear path to a solution.***

Using this definition, "Problem" 1, above is not a problem for most children who have demonstrated proficiency in addition and understand when addition is to be used. In fact, the way some "problem-solving" units are placed in chapters in mathematics textbooks (at the end of chapters following computational drill on the operation required in the verbal exercises), the problems are simply verbally-stated exercises. On the other hand, for a typical kindergarten student, Problem 1 presents a situation that might involve problem solving. The child, after hearing and understanding the situation, might use counters to represent and solve the problem.

The situations in Problems 2 and 3 could be classified as nonroutine problems. For most students, the path to a solution for these situations would not be immediately apparent. On the other hand, the solution is within reach of most middle and upper elementary students. If a student were to actively engage himself in finding a solution for these, he would be involved in problem solving.

Whether or not a situation involves problem solving, then, depends on the way the problem is stated, the context of the problem, and the ability level and involvement of the learner. Problems may involve applications of mathematics (e.g., calculating how much items cost that are on sale at a 25% discount), computations (e.g., dividing a number by another with a remainder), or mathematical principles (e.g., discovering the Pythagorean Theorem). Whenever the learner does not have an algorithm to help her obtain a quick solution to a mathematical situation, and assuming that she makes an attempt to find a solution, she is involved in problem solving.

TEACHING PROBLEM SOLVING

The trend in modern mathematics textbooks is to include many problems, both traditional and nontraditional. Furthermore, *The Arithmetic Teacher,* other journals in the field, and many resource books for teaching mathematics (e.g., Kruglick and Rudnick, *Problem Solving: A Handbook for Teachers,* 1980) include a large variety of problems requiring different resources and mathematical skills. A list of resources for teaching problem solving is given at the end of this chapter.

Compared to drill and practice on computational techniques, solving problems is a lot of work. For the student, there is (by definition) the uncertainty of not knowing what strategy to use to find a solution. For the teacher, there is the uncertainty of not knowing how to improve students' problem-solving capabilities. However, certain methods improve problem-solving capabilities of students and thus make problem solving a less frustrating task for all. The rest of this chapter covers these methods.

One of the most important suggestions for improving problem-solving performance is to increase the number and variety of problems to which children are exposed. Research has shown that as children are given more problem-solving experience, their ability to solve problems increases (Suydam 1982). In order to make children better problem solvers, we should expose them to traditional word problems, applied problems, and nontraditional problems. Not only will their problem-solving performance be improved but they will find mathematics to be more interesting and relevant.

An equally important suggestion is to help children develop successful strategies for solving problems. One of the first people to address the area of problem solving was George Polya of Stanford University. He wrote a classical book called *How to Solve It* (1957) that outlines a strategy to attack and solve problems. The following statements paraphrase the steps in the strategy he recommended.

1. Read and try to understand the problem.
2. Develop a plan for solving the problem.
3. Carry out the plan.
4. Check your answer to see if it makes sense.

Other mathematics educators and researchers have elaborated upon this list. They have identified more specific problem-solving strategies that improve childrens' ability to find solutions to problems. Each of the steps in the problem-solving process that Polya listed is discussed below.

Understanding the Problem

The goal here is to have students understand the problem situation: that is, what is being asked for in the problem. Several approaches have been shown to help students focus on what the problem situation is.

A. Have students restate the problem in their own words before attempting a solution. This will force the student to read and understand what the problem is asking rather than diving in and doing the first computational procedure that comes to mind. It is also suggested that students occasionally be given sets of problems where they do nothing but restate the problem (without doing any computation at all). The teacher then collects and evaluates their work to see if they have understood the problem situation.

PROBLEM: The Yellow Pizza Music Shop is having a sale. All records are on sale at $2.00 off and all tapes are on sale for $1.50 off. Monica decided to buy two records marked $7.95 and $8.45 and a tape marked $6.50. Not counting the sales tax, how much did she pay?

Restate in your own words:
The Yellow Pizza Record Shop is selling records at $2 off and tapes at $1.50 off. Monica bought two records and a tape. The records cost $7.95 and $8.45 and the tape cost $6.50. How much did she pay?

B. Have students make a list of everything they know about the problem. This is similar to A above, but does not require complete sentences and may include things they have surmised about the problem that were not stated.

PROBLEM: School Bus 5 picks up 5 children at its first stop, 8 at the second stop, 3 at the third stop, 8 at the fourth, 6 at the fifth, 1 at the sixth, and 4 at the seventh. The eighth stop is at the school where all the children get off. How many children did the bus let off at the school?

List everything you know about the problem:

Stop 1—5 students
Stop 2—8 students
Stop 3—3 students
Stop 4—8 students
Stop 5—6 students
Stop 6—1 student
Stop 7—4 students
Stop 8—All get off
How many get off?

C. Represent the problem. This may be done using objects like toys or play money. Or, it may involve acting out a situation together with other pupils. For example in Problem 1 above, one pupil could play the role of Susie and another the role of John. They would combine the books and count to find the answer. Another way to represent a problem is to use diagrams or pictures. Drawings help a child solve a problem involving the spatial arrangement of objects (e.g., Johnny is taller than Susan but shorter than Beverly), while flowcharts are helpful in finding answers to situations that involve the sequencing of events.

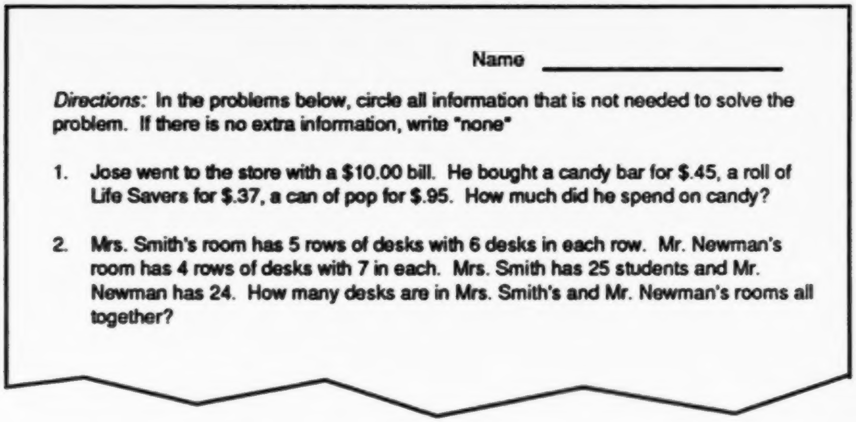
D. For poor readers, record the problem on tape and have them read while the tape is played. Explain new vocabulary to them. Have children read the problems to each other.

E. Have students identify any unnecessary information in the problem. Occasionally you may wish to give students problems that contain extraneous information and simply have them put a line through information in the problem that is not needed to solve it. Next, give them problems with both necessary and unnecessary information and have them identify the information needed to solve the problem. Their work would be evaluated on their ability to identify and distinguish extraneous from needed information. Of course, the final step would be to have children efficiently find answers to problems containing both necessary and unnecessary information. (See Figure 5.1.)

Develop a Plan

After the student has understood the problem situation, the next step is to develop some systematic approach for finding a solution. In simple one-step problems, this may only involve identifying the operation to be carried out. In others, like Problems 4 and 5 at the beginning of this chapter, it may involve identifying a series of operations. It may also involve identifying other resources like maps or computer software.

FIGURE 5.1
A worksheet for identifying unnecessary information.



Name ____________________

Directions: In the problems below, circle all information that is not needed to solve the problem. If there is no extra information, write "none"

1. Jose went to the store with a $10.00 bill. He bought a candy bar for $.45, a roll of Life Savers for $.37, a can of pop for $.95. How much did he spend on candy?
2. Mrs. Smith's room has 5 rows of desks with 6 desks in each row. Mr. Newman's room has 4 rows of desks with 7 in each. Mrs. Smith has 25 students and Mr. Newman has 24. How many desks are in Mrs. Smith's and Mr. Newman's rooms all together?

A. Have students identify the operation(s) needed to solve the problem. Occasionally, you may wish to give students a set of 10 or so problems where they simply write the operation or sequence of operations to use in the solution without doing any computation. Also, you may want to give them problems that cannot be solved because they don't provide the complete numbers, but that include enough information so that a *plan* for a solution can be developed. For example:

PROBLEM: Julie had $20 in her purse when she went into the discount store. She bought a roll of film, a tube of toothpaste, and a pair of earrings. If the sales tax is 5% in her city, how much money did she have left when she left the store?

How would you solve this problem (what operations would you do)?

B. Have the students decide if a number sentence will help solve the problem. Certain problems lend themselves to solution through number sentences while for others a table or diagram is more appropriate. For example:

PROBLEM: A shirt marked at $27.00 is being sold for ⅓ off. What is the sales price?

Mark the best way to solve this problem:

Number Sentence ______ Table ______ Diagram ______ Graph ______

C. Estimate the answer to the problem before calculating. This helps focus attention on the type of strategy that could be used and helps the student determine if the strategy she envisions will yield appropriate results. Sometimes you may have the student write down her estimate so that she may compare it to the answer when she obtains it to see if it is reasonable. For example:

PROBLEM: The auditorium at Chayfee School has 27 rows of seats with 31 seats in each row. How many people will the auditorium seat?

What operation should you use to find an answer? ____________________

Estimate the answer to this problem: ____________________.

Next, the pupil would carry out the plan and compare the computed solution to the estimate.

D. Determine if a table, graph, or diagram would be helpful in organizing the data in the problem so that it can be solved. For example:

PROBLEM: Mr. Paul's science class decided to see what effect temperature has on the activity level of snails. They gathered 20 snails and found the average distance they moved in 5 minutes at five different temperatures. The results were: 90 degrees—15 cm, 50 degrees—0 cm, 70 degrees—4 cm, 60 degrees—3 cm, 80 degrees—7 cm.

Which would be most useful in determining the effect of temperature on the activity level of snails?

Number Sentence ______ Table ______ Diagram ______ Graph ______

E. Identify any resources that might be needed to help solve the problem. Problem 4 at the beginning of the chapter might be best solved using a calculator or a computer. Problem 5 will need maps, airline schedules, gasoline prices, train schedules, and so forth.

Carry Out the Plan

This is the stage where the student gathers data, analyzes it, computes, and if the results are not reasonable, goes back to the drawing board and develops a new plan. The flow diagram in Figure 5.2 illustrates the entire problem-solving procedure.

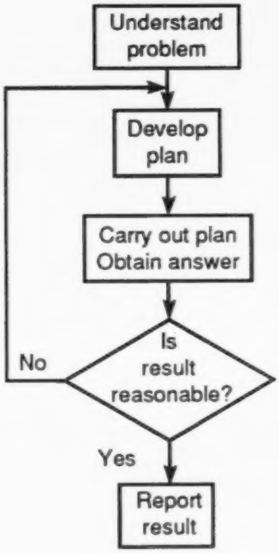


FIGURE 5.2
A problem-solving flowchart.

Some of the things that can be done to help solve the problem are:

A. Make an organized list or table. This strategy is particularly helpful when information is categorical in nature. Consider the following problem for example:

PROBLEM: The third grade class in Thomas School has 3 classes taught by Mrs. Smith, Mr. Sanchez, and Ms. Jones. Mrs. Smith's class has 14 boys and 16 girls. Mr. Sanchez's class has 17 boys and 31 total students. Ms. Jones class has 12 girls and 26 total students. How many boys are there in the third grade at Thomas School? How many girls? How many third graders are there?

SOLUTION:

	Boys	Girls	Total
Mrs. Smith			
Mr. Sanchez			
Ms. Jones			
Total			

B. Make a diagram. A diagram is a line drawing made to represent a situation in simplified form in order to help understand it. The following problem is an example:

PROBLEM: Ashe street runs East and West. Alice, Beth, Carl, and Doris all live in different houses on the same side of Ashe street. Their house numbers are 28, 32, 34, and 36, but not necessarily in that order. Alice and Carl live to the East of Doris. Beth's house number is higher than Doris'. Carl's house number is the lowest. What are the street numbers of each person's house? (See Figure 5.3.)

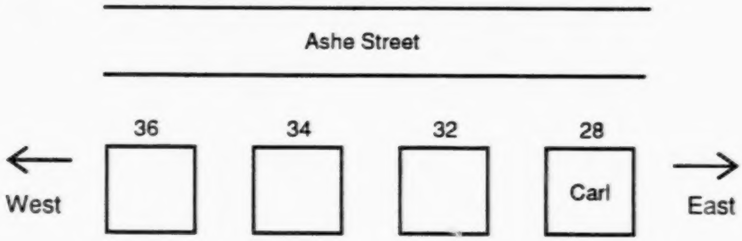


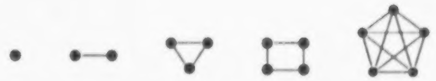
FIGURE 5.3
Drawing a diagram.

C. Look for patterns. Many mathematical problems can be solved by observing systematic increases or decreases in the numerical values generated in the problem. In order to train students to look for these patterns, give them appropriate problems and have them record the numbers and examine the results for patterns that might lead to a solution.

PROBLEM: In a round-robin ping-pong tournament (everyone plays everyone else in the tournament exactly once), there were 36 total matches. How many participants were in the tournament?

SOLUTION: To solve this problem it is helpful to draw diagrams, starting with 1, 2, 3, etc. participants, then look for a pattern in the numbers generated. (See Figure 5.4.)

FIGURE 5.4
Looking for patterns.



Players	1	2	3	4	5	6	7	...	...	
Matches	0	1	3	6	10	?	?	?	?	

D. Solve a similar but easier problem. Often times problems seem complex because of the values used. In reality the solution technique may be quite simple. Finding a parallel, but simpler problem may lead to a solution in the more complex problem.

PROBLEM: A video game is on sale in a department store for 25% off. The original price is $24. What is the reduced price?

PARALLEL PROBLEM: An item is sold in the store for $2 off. If the original price of the item was $10, what is the sales price?

SOLUTION: In the parallel problem, $2 is subtracted from $10 to get the sales price. In the original problem, the discount amount is subtracted from $24 to get the sales price. The discount is 25%. To find the discount in dollars, multiply 25% by $24.

E. Write and solve the number sentence. Often, this will help the learner identify the operations needed and the order in which they are performed to solve the problem, especially if more than one operation is needed.

PROBLEM: Julie went to the store with a $10.00 bill. She bought a record for $5.98 and an ice cream cone for $.45. On the way home, she found a quarter. How much money did she have when she arrived home?

SOLUTION: $10.00 − 5.98 − .45 + .25 =

Other Techniques

A. Allow children to use calculators to help them solve problems. The purpose of teaching problem solving is not to force children to practice computation techniques, but rather to help them become better problem solvers. A study by Wheatly (1980) found that children who use calculators in problem solving have more bright ideas, try more paths to a solution, and are more successful than children who use pencil and paper. The use of the calculator allows you to give children more realistic and interesting problems to solve. It allows them to attack more problems in a given time. Calculators also tend to motivate children by their very presence. If calculators are used no other place in the classroom, they should be used in the problem-solving context. Problems 4, 5, and 6 at the beginning of this chapter are examples where calculators would be useful. In 4 and 5, they might be essential.

B. Where appropriate, allow students to use computers to help them solve problems. For those who have had programming experience, certain problems (like Problem 4 at the beginning of the chapter) might lend themselves to a programmed solution. Certain computer programs like statistical packages, spread sheets, or file programs are helpful. For example, if students do an integrated math/science project where they record the growth of plants in different light and soil conditions, they could record the data using a spreadsheet and make predictions about growth in a variety of conditions.

C. Include problems that require a variety of resources and take considerable time to complete. Too often we condition students to expect immediate results from any task attempted. Problem 5 (presented at the beginning of this chapter) is an example of a situation whose solution should take time and effort, and the use of a broad range of resources.

Check Answer

Sometimes children are so answer oriented that once they have obtained an answer, they rush on to the next problem without stopping to ask whether the answer is reasonable. An important step in the problem-solving process is to examine the answer obtained to see if it makes sense. If not, the problem solver needs to return to either step 1 (understand the problem) or step 2 (develop a plan) and proceed from there.

The teacher can do a number of things to encourage children to examine the reasonableness of their answers.

A. Ask children to specifically state whether their answer makes sense.

PROBLEM: Johnny grew $1 \frac{1}{2}$ inches when he was eight years old, $\frac{3}{4}$ inches when he was nine, and $2 \frac{1}{8}$ inches when he was ten. How much did he grow from the age of eight to the age of ten?

Use appropriate strategies to solve this problem (write each step).

Your answer:

Is this answer reasonable?
Why not?

If the answer is not reasonable, solve the problem again. List each strategy you use and check to see if your new answer is reasonable.

B. Another possibility is to ask the problem solver to estimate the answer to the problem before doing any computations, then to compare the calculated answer to the estimate. If the answer is substantially different from the estimate, the child should ask herself whether the estimate was inappropriate or whether she should re-solve the problem. The process of estimating answers, calculating answers, and then comparing the two not only results in more frequent correct answers, but also improves children's estimation abilities.

C. A final possibility is to have children compare their computed answers to correct answers supplied by the teacher, then, if their answer is different, resolve the problem. Although it is more desirable to develop in children the ability to evaluate the reasonableness of their own answers rather than always looking to the teacher or textbook for the correct answer, children (especially novice problem solvers) do not always have this capability. In the beginning stages of problem solving, or when facing non-routine problems or those involving new operations or computational techniques, students need some sort of feedback as to the reasonableness of their answers. This feedback should not come in the form of a grade on the top of a homework paper, but should be given while students are working on problems to help them decide if they need to go back to step one or two of Polya's guidelines.

The teacher's input to the problem-solving process need not be limited to giving the correct answer. It may include telling the student that her answer is too large or small, that the correct answer is between 25 and 100, or that she might want to try using a table or diagram to solve the problem. We want to encourage students to try a variety of strategies and to move toward more reasonable results. Therefore, we should ask them to record the steps and strategies they use, even the inappropriate ones. Not only does this allow us to follow children's thinking processes, but it gives them the message that the *process* of solving problems is as important as the *product*!

A Final Note

Problem solving is quite a different task from computation. When a student has learned a computational technique, he should be able to repeat the algorithm successfully on demand. It is reasonable, then, to expect the student to score 80 or 90 percent on a test or drill sheet covering that task. Problem solving, on the other hand, is a completely different kind of endeavor. Normally, we do not expect students to solve true problems under time constraints. Such an expectation would be extremely frustrating for students. In addition, it might be reasonable for us to expect a student to only successfully solve only 33% of problems given. In other problem-solving situations, the student could be given credit if he successfully restates the problem and sets up a plan for solving it. A problem-solving atmosphere calls for flexibility on the part of the teacher—flexibility in terms of requirements, movement and interaction of students, and the varied approaches and answers of different students.

SUGGESTED READINGS

Arithmetic Teacher 29(6), February 1982. Problem solving is the theme of this issue. Ten articles are about problem solving.

Carpenter, T. P., et al. Solving verbal problems: Results and implications from National Assessment. *Arithmetic Teacher* 28(1), September 1980:8–12.

Charles, R. I. Get the most out of 'word problems.' *Arithmetic Teacher* 29(3), November 1981: 39–40.

Davidson, J. E. The language experience approach to story problems. *Arithmetic Teacher* 25(1), September 1977: 28.

DeVault, M. V. Doing mathematics is problem solving. *Arithmetic Teacher* 28(8), April 1981: 40–43.

Greenes, C. E., and Schulman, L. Developing problem-solving ability with multiple-condition problems. *Arithmetic Teacher* 30(2), October 1982: 18–21.

Greenes, C. E., Immerzeel, Ockenga, Schulman, L., and Spungin. *TOPS Story Problems (Addition and Subtraction, Multiplication and Division, One- and Two-Step Problems, Guess and Check, Logical Thinking, Using Resources).* Palo Alto: Dale Seymour Publications, 1982.

Johnson, D. C. Teaching estimation and reasonableness of results. *Arithmetic Teacher* 27(1), September 1979: 34–35.

Kilpatrick, J. Stop the bandwagon, I want off. *Arithmetic Teacher* 28(8), April 1981: 2.

Krulik, S., and Rudnick, J. A. *Problem Solving: A Handbook for Teachers.* Boston: Allyn and Bacon, 1980.

Moses, B. Individual differences in problem solving. *Arithmetic Teacher* 30(4), 1982: 10–14.

Polya, G. *How to Solve It.* Garden City, NY: Doubleday Anchor Books, 1957.

Problem Solving in School Mathematics (1980 Yearbook). Reston, VA: National Council of Teachers of Mathematics, 1980.

Suydam, M. N. Update on research on problem solving: Implications for classroom teaching. *Arithmetic Teacher* 29(6), February 1982:56–60.

Wirtz, R. W., and Kahn, E. Another look at applications in elementary school mathematics. *Arithmetic Teacher* 30(1) September 1982:21–25.

Yeager, D. C. *Story Problems.* Palo Alto, Calif.: Creative Publications, 1982.

______. *The Kennedy Kids* (Books 1, 2, and 3). Palo Alto: Creative Publications, 1985.

QUESTIONS

1. Use the strategies above to solve the following problems—write out all strategies you use:
   a. What is the value of all the gold in the world if (when it was all melted down and formed into a cube) it would make a 10 by 10 by 10 yard cube? Use today's gold price.
   b. How many squares can you make on a geoboard?
   c. The new school has exactly 1000 lockers and exactly 1000 students. On the first day of school, the students meet outside the building and agree on the following plan: The first student will enter the school and open all of the lockers. The second student will then enter the school and close every locker with an even number (2, 4, 6, 8, . . .). The third student will then "reverse" every third locker. That is, if the locker is closed, he will open it; if it is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 1000 students in turn have entered the building and reversed the proper lockers. Which lockers will finally remain open? (Krulik and Rudnik, 1980)
2. Find a recent third grade math textbook and accompanying teacher's edition. Look for a "problem-solving" section in the book. Discuss whether the section really deals with problem solving or not.
3. Describe a problem-solving activity for children involving a local retail store.
4. Discuss whether or not problem-solving questions should be included in standardized tests.

CHAPTER

6

Estimation & Mental Arithmetic

You are checking out at the grocery store and thinking about the TV movie you abandoned for this shopping errand. You absentmindedly place a half-gallon of milk, a loaf of bread, and a jar of mayonnaise on the counter. The clerk enters the items into the cash register and says, "That will be twelve dollars and eighty-five cents please." You jump to attention, staring at the clerk with your mouth open and protest, "There must be some kind of mistake; these things should not cost more than four dollars." The clerk nods her head and smiles as she checks the printout. "Sorry," she says, "I guess I entered the wrong amount for the mayonnaise."

That evening you take a friend out to dinner. The bill at the end of the meal is $24.86. In order to determine what to write on the charge slip as a tip, you mentally round the bill to $25.00 and move the decimal over one place to the left to get $2.50. Next, you determine that half of $2.50 is $1.25 and add the two amounts together (mentally, from left to right) to get $3.75 for the tip. You write this amount on the charge card slip, add the tip to the price of the meal, and sign it, confident that your mental computation is accurate.

That evening you are balancing your checkbook with your calculator. At one point, your calculator registers a negative balance of $597.80. Since your previous entry showed a positive balance of over a hundred dollars and since the check you just entered was for $60.00, you realize that you should still have a positive balance and that you must have entered $600.00 rather than $60.00.

The above situations are typical of our everyday use of mathematics. In fact, most of us probably do this type of math more frequently than we do traditional pencil-and-paper computations. This is especially true in this era of technology: we rely on calculators, computers, and sophisticated cash registers when exact computations are needed involving large and/or many numbers, but at other times we use mental computation techniques to quickly arrive at approximate answers. In rare instances, we (reluctantly) drag out pencil and paper and do the computations by hand.

Despite the importance of mental arithmetic and estimation, it is a subject which, until recently, has rarely been covered in the regular school curriculum. In 1984 Marilyn Suydam stated:

> Textbook analyses indicate that it has been one of the most neglected topics in the curriculum. It is therefore no surprise that students perform poorly on estimation questions in national assessments. When asked to estimate the answer to $^{12}/_{13} + {}^{7}/_{8}$, only 24 percent of the thirteen-year-olds queried could do so correctly; when asked to estimate the answer to 3.04×5.3, only 21 percent could do so. These percentages are just above the chance level. (35)

This is changing. Many of the newer K–8 mathematics textbook series include some instruction and practice in mental arithmetic and estimation. Of course the inclusion of a topic in a mathematics textbook does not assure that it will be taught to or mastered by a student. In order to help our students be successful in mathematical situations, schools need to place more emphasis on estimation and mental arithmetic—they should be included as a standard part of the curriculum. We need to teach children the best techniques for determining if the answers they obtain with their calculators, cash registers, and computers are correct. We also need to teach them techniques for obtaining quick estimates in computational situations when technological device or pencil and paper are unavailable or inappropriate.

In each of the estimation situations presented at the beginning of this chapter, the computations were done mentally, without pencil and paper; answers were needed quickly; and the answers obtained did not have to be exact. Of course, sometimes estimation is *not* appropriate, like when you fill out your federal income tax forms: imagine telling the IRS that you have calculated the amount of taxes you owe to the nearest $500. On the other hand, computational estimation is essential as you compute and check the amount you owe the government and as you complete the forms.

Estimation is not done in a vacuum. Success in estimation, even more than success in pencil and paper computation, depends on an understanding of our numeration system, on the ability to mentally find the answers to single-digit computations (the "basic facts"), and on an understanding of mathematical principles such as the commutative property or the meaning of multiplication.

How do we teach children to arrive at appropriate estimates to computational problems? The procedures that we can teach students include converting the values in the problem to those that can be solved through a combination of basic facts, moving a decimal, or doubling or halving a quantity. Prerequisite to estimation skills are fundamental math concepts, immediate recall of basic facts, and thorough understanding of place value and numeration since most mental arithmetic involves rounding, truncating, or changing the problem values to adjacent values.

SETTING THE STAGE

Teaching estimation is, in a sense, very similar to teaching problem solving—it involves a fair amount of uncertainty. Unlike computational algorithms where most people have been taught the same techniques for arriving at exact answers, several people asked to estimate the answer to the same problem will use a variety of strategies and get varying results, some of which are acceptable, others of which are not acceptable. For children who have been led to believe that there is one and only one correct answer for a math problem and one way of obtaining that answer, the thought of a variety of approaches and possible answers will probably be unsettling. As we do when we are teaching problem solving, we need to help children deal with the prospect of selecting from a number of possible strategies and of determining the accuracy and acceptability of their answer.

Teachers need to create an atmosphere in the classroom where a variety of approaches and answers are accepted. This may be done by encouraging students to share their approaches and answers, and by accepting these approaches while leading them to use better, more efficient ones. Further, students need to engage in computational estimation often and early—as soon as work with multi-digit computation is begun. As with problem solving, the more estimation is done, the more proficient children become.

Important conceptual prerequisites for mental computation are rounding and truncation. These, in turn, are based on a firm understanding of place value and other numeration concepts. Teachers should make sure that students have a grasp of these concepts and can accurately round numbers to given places. We also need to lead students to understand why we focus on front-end digits when doing mental arithmetic. This may be done by selecting a variety of applied numbers (e.g., "there are 397 children in the school" or "I have $158 in my savings account") and discussing which of the digits in these numbers tells us the most about how many there are.

LEFT-TO-RIGHT COMPUTATION

Our usual way of doing whole number computations is from the least to the most significant digits—right-to-left. Our numeration system, however, is *left-to-right*—we read and write numbers in that direction and we tend to remember and focus on the most significant digits (the left-most) in the numbers we hear.

The computational algorithms we usually use move right-to-left because we regroup from smaller places to larger places when we add. If we go from left-to-right, we have to check the results of the next step(s) to the right to complete an operation in a place. When we add or subtract just two numbers, however, it is quite easy to start with the most significant (left-most) place and check the next place to the right to see if we should increase or reduce our sum. In fact, many people (including myself) occasionally add or subtract from left to right using this technique. (See Figure 6.1.) Chapters 10 and 11 include discussions of methods for teaching left-right addition and subtraction.

FRONT SIGNIFICANT DIGITS

One of the most important ways of handling computational situations mentally is to focus students' attention on the most significant digits of the numbers being operated on. For example, if you were asked to estimate the cost of four record albums priced at $6.15 apiece, you would multiply six dollars by four to get $24.00,

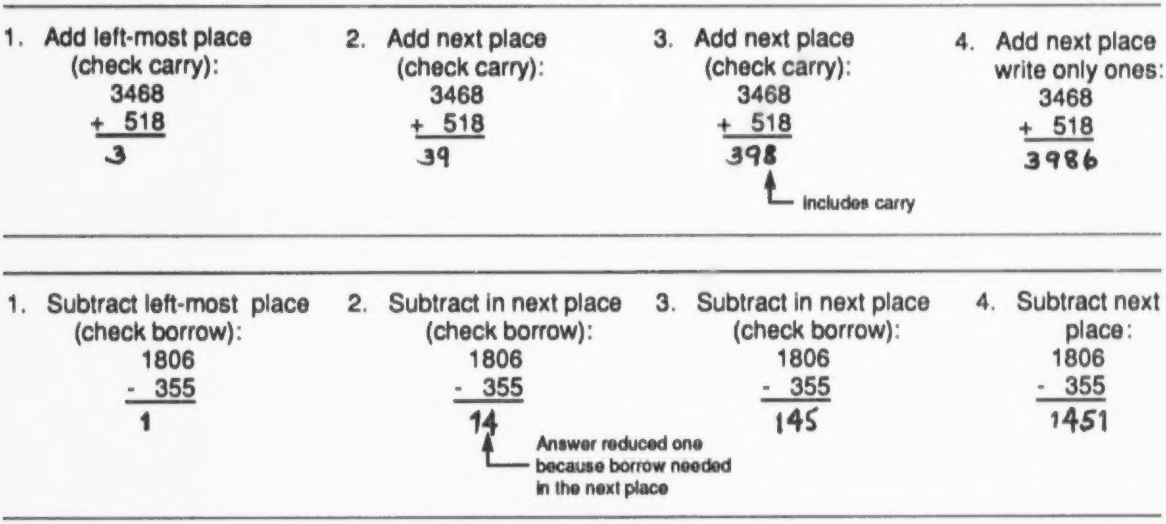

1. Add left-most place (check carry):
3468
+ 518
3
2. Add next place (check carry):
3468
+ 518
39
3. Add next place (check carry):
3468
+ 518
398
includes carry
4. Add next place write only ones:
3468
+ 518
3986
1. Subtract left-most place (check borrow):
1806
- 355
1
2. Subtract in next place (check borrow):
1806
- 355
14
Answer reduced one because borrow needed in the next place
3. Subtract in next place (check borrow):
1806
- 355
145
4. Subtract next place:
1806
- 355
1451

FIGURE 6.1
Left-to-right addition and subtraction.

then perhaps adjust the answer upward to $25.00. If you were asked to estimate the combined population of three adjacent towns with populations of 4215, 7306, and 225, you would focus on the 4 (4000) and 7 (7000) in the first two numbers, (of course ignoring the 2 in 225 since its place value is different from the 4 and 7) and conclude that the population is more than 11 thousand. Suppose that you are one of five people doing a job and the $630 earned is to be divided equally among the workers. If you want to quickly estimate how much money you could spend on a Christmas gift for your mother, you might mentally divide 600 by five and conclude that you will earn more than $100. For example:

Step 1: $630 ≈ $600 (Round)
Step 2: 6 ÷ 5 is larger than 1
Step 3: The place value of 1 is 100, thus the answer is more than $100.

Whenever we do front-end mental operations, we need to be aware of the place values of the front digits and also the values of the digits immediately to the right of the front digits. Consider these two sums, for example:

$$\begin{array}{rr} 286 & 412 \\ 475 & 300 \\ 188 & 221 \\ \underline{+392} & \underline{+117} \end{array}$$

By focusing on the front digits only, we get the same estimate for these two problems—1000. However, if we consider the digit that is second from the front we get 1300 for the first and 1000 for the second. Therefore after we have done our computation using the front-end digit, we scan through the digits immediately to the right and make any needed adjustments. (See Figure 6.2.)

The front-end approach to mental computation should be the first kind introduced to children because it is the easiest to master. According to Robert Reys,

> the advantage of the front-end strategy over the traditional rounding approach is that all the numbers to be operated on are visible in the original

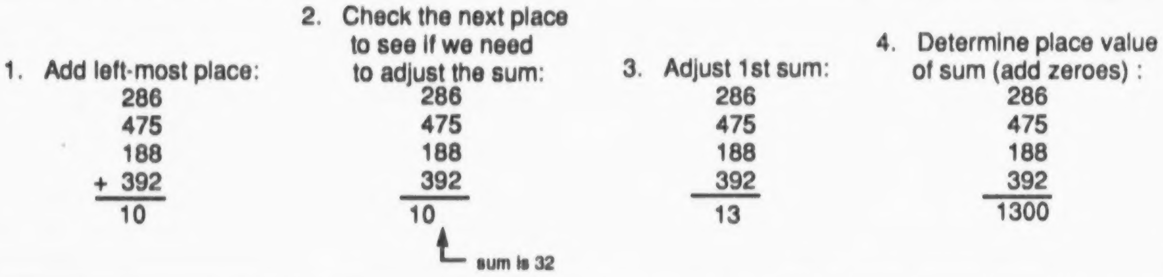

1. Add left-most place:
286
475
188
+ 392
10
2. Check the next place to see if we need to adjust the sum:
286
475
188
392
10
sum is 32
3. Adjust 1st sum:
286
475
188
392
13
4. Determine place value of sum (add zeroes) :
286
475
188
392
1300

FIGURE 6.2
Front-end addition.

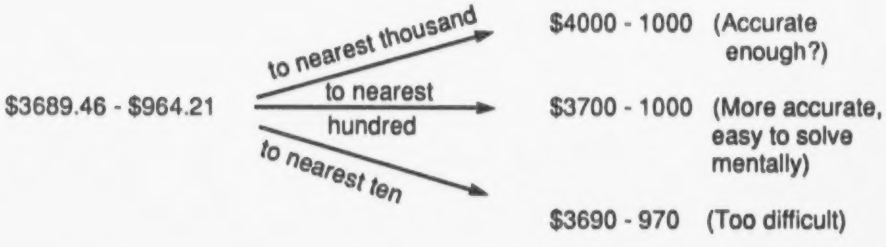
problem. It gives even young students a way to arrive at an estimate quickly and easily. This feeling of success ("I can do it") is important. Although this strategy is especially good for young students, it is also perfectly appropriate and useful for older students and adults. (1984b, 1)

ROUNDING/COMPATIBLE NUMBERS

The process of rounding followed by mental addition, subtraction, multiplication, or division is similar in many ways to front-end computation. However, whereas front-end computation is straightforward, the rounding approach involves numerous decisions—how many digits am I going to consider, what kind of rounding will make the problem mentally solvable, what kind of rounding will give me an accurate answer? (See Figure 6.3.)

One of the biggest challenges in using the rounding technique is deciding what values to round to so that the problem is solvable through mental arithmetic. That usually means that the numbers are simplified to the point where only the most significant digit is non-zero, and/or the values we round to are solvable through our knowledge of basic facts (i.e., the numbers are compatible) and by moving decimals. For example, if you want to estimate how much money you would make a day if you are offered a job for $242.45 for a 5 day work week, you round $242.45 to $250 because you know that it is divisible by 5.

FIGURE 6.3
Rounding.


$3689.46 - $964.21
to nearest thousand
to nearest hundred
to nearest ten
$4000 - 1000 (Accurate enough?)
$3700 - 1000 (More accurate, easy to solve mentally)
$3690 - 970 (Too difficult)

The mental search in the above problem might go as follows:

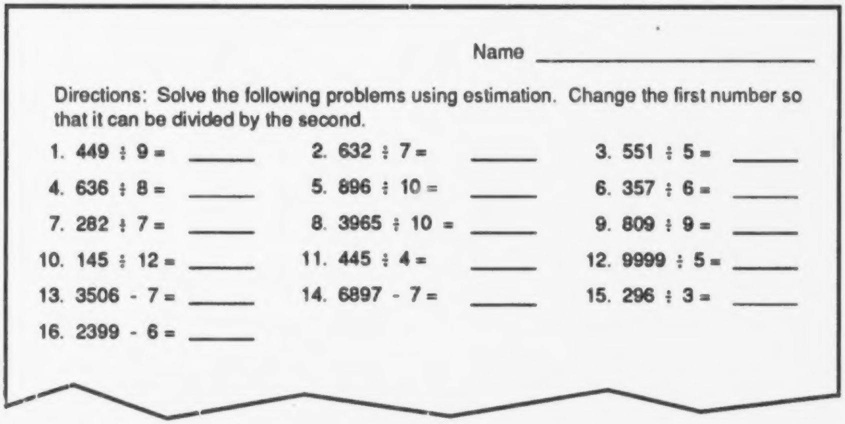
Step 1: Rounding \$242.45 to nearest 100 gives \$200. This will not give me a very accurate result.

Step 2: Rounding \$242.45 to nearest 10 gives \$240. This is not easily divided by 5.

Step 3: \$242.45 is close to \$250, which can easily be divided by 5!

When teaching students to look for compatible numbers, first give them appropriate examples—those where the rounding strategy that produces compatible operands is fairly obvious. Have students find answers and then discuss their strategies with them one at a time. Some examples of problems in which the change to compatibles is fairly obvious are shown in Figure 6.4. Next, move to problems where the compatibles are not as obvious or where there are several possibilities—problems like 23 × 172 and 2476 ÷ 17 are examples.

FIGURE 6.4
Estimation with obviously compatible numbers.



Name ______________________

Directions: Solve the following problems using estimation. Change the first number so that it can be divided by the second.

1. 449 ÷ 9 = ______ 2. 632 ÷ 7 = ______ 3. 551 ÷ 5 = ______
4. 636 ÷ 8 = ______ 5. 896 ÷ 10 = ______ 6. 357 ÷ 6 = ______
7. 282 ÷ 7 = ______ 8. 3965 ÷ 10 = ______ 9. 809 ÷ 9 = ______
10. 145 ÷ 12 = ______ 11. 445 ÷ 4 = ______ 12. 9999 ÷ 5 = ______
13. 3506 - 7 = ______ 14. 6897 - 7 = ______ 15. 296 ÷ 3 = ______
16. 2399 - 6 = ______

The second important challenge is rounding the values in the problem so that the answer you obtain is accurate. If you want to estimate the number of students in a school and you know that there are 18 home rooms with about 26 pupils in each, the strategy of looking at each number in isolation without considering the type of operation (and therefore rounding each number up):

$$18 \rightarrow 20 \qquad 26 \rightarrow 30 \qquad 20 \times 30 = 600$$

would produce poorer results than if, realizing the nature of multiplication, you rounded 18 up to 20 and 26 down to 20:

$$18 \rightarrow 20 \qquad 26 \rightarrow 20 \qquad 20 \times 20 = 400$$

or better still, rounded 18 to 20 and 26 to 25:

$$18 \rightarrow 20 \qquad 26 \rightarrow 25 \qquad 20 \times 25 = 500$$

Students should, therefore, keep the nature of the operation involved in mind during the operation.

When we teach the rounding technique for estimation, not only do we need to teach specific techniques, but we also need to teach flexibility—the ability to mentally examine several alternatives, decide which is the best, compute, and then revise the answer, if necessary.

After rounding, the values in a computational situation are no longer visible (assuming the problem was originally written). Not being able to see the rounded numbers can create an added difficulty for the problem-solver. Often a first step in the process of developing rounding/estimation ability is to have students round the values in the problem, then write them down before continuing with the mental operation.

Once students obtain an answer, they should evaluate it. Is their estimate reasonably close to the exact answer? Students can use a calculator to do the exact computation and compare the estimated answer to the calculated answer. See Figure 6.5 for a worksheet involving estimation problems.

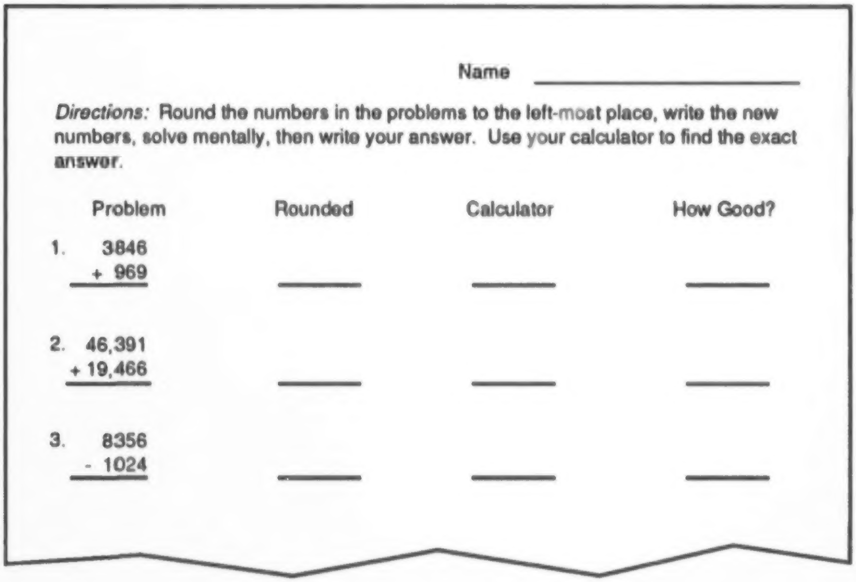


FIGURE 6.5
An estimation worksheet.

Name ______________________

Directions: Round the numbers in the problems to the left-most place, write the new numbers, solve mentally, then write your answer. Use your calculator to find the exact answer.

Problem	Rounded	Calculator	How Good?
1. 3846 + 969	______	______	______
2. 46,391 + 19,466	______	______	______
3. 8356 - 1024	______	______	______

FRACTIONS AND PERCENTAGES

When working with fractions or percentages, the most useful strategies involve converting the adjacent values that make the problem solveable, or changing the problem to another type of operation. For example, if you needed to approximate $\frac{2}{3} \times \frac{5}{8}$ (suppose, for example that you are reducing a recipe calling for $\frac{5}{8}$ of a stick of margarine), you might "round" $\frac{5}{8}$ to $\frac{1}{2}$ then find half of $\frac{2}{3}$ and conclude that the answer is about $\frac{1}{3}$ of the stick. If we add fractions, we want to change the fractions to adjacent fractions with like denominators, allowing easy mental addition. For example:

$$\frac{5}{8} + \frac{1}{3} \rightarrow \frac{5}{8} + \frac{3}{8} \rightarrow \frac{8}{8} = 1$$

When multiplying fractions, we convert each of the values in the problem to an adjacent whole number or a half, a third, or a fourth, allowing us to multiply by the whole number or to divide by 2, 3, or 4, respectively. For example:

$$\frac{2}{7} \times 15 = ? \rightarrow \frac{1}{3} \times 15 = 5$$

In the case of mixed fractions, problems may be rounded to a whole number so that the fraction is eliminated. For example:

$$10\frac{1}{2} \times 43 \rightarrow 10 \times 43 = 430$$

When dealing with percentages, the most useful strategies involve changing the problem to a similar problem that can be solved through whole-number multiplication or division, or by moving a decimal. For example, 48% of 120 can be rounded to 50% of 120, then solved mentally by dividing 120 by 2. If you wish to leave a 10% tip in a restaurant where the bill is $24.56, you might first round the bill to $24 to facilitate the mental computation.

AVERAGING

Certain types of operations are such that it is possible to switch to easier operations. For example, if you wanted to estimate the number of students in the fourth grade at Fremont School and the exact class counts were as follows:

Class	*Count*
Mrs. Anderson (Rm. 5)	34
Mr. Chavez (Rm. 6)	29
Mrs. Washington (Rm. 9)	32
Mrs. Wiederhold (Rm. 23)	28
Mr. Francona (Rm. 3)	31

A quick glance at the class counts shows that all the numbers are close to 30. If we estimate that the *average* count is 30, we can multiply 30 by the number of classes—5—to get an estimate of 150 students.

To teach this type of mental arithmetic, give examples of this type of problem and discuss the basic strategy, then given students problems that fit into the limitation of this kind of strategy. The limitation here is that the problem involves addition of several numbers of relatively the same size.

COMPENSATION

Compensation is the process of adjusting an estimated answer after the mental computation, to make up for initial rounding or truncation. For example, if you needed to estimate the product of 290 and 93, you might round 290 to 300 and 93 to 100, multiply, then subtract something from 30,000 to compensate for rounding both the original numbers up. Since we added 7 to 93 when we rounded it to 100, we know that our total (30,000) is at least 7 × 300 (the number we rounded 290 to) or 2100 too large. Since we also added something to 290 when we rounded it up, we might want to increase the amount we subtract from 30,000 to 3000, giving us 27,000. This gives us a very good estimate of the product of 290 and 93 (the exact product is 26,970).

When compensating, the amount of adjustment done depends on the operation performed. When multiplying or adding, if both numbers operated on are both rounded down, the resulting value might be adjusted upward whereas the answer would be adjusted downward if both original numbers were rounded up. For example:

$$\begin{array}{ccccc} \textit{Old Problem} & & \textit{New Problem} & & \textit{Adjustment} \\ 4166 & & 4000 & & \\ \underline{+7098} & \rightarrow & \underline{+7000} & \rightarrow & 11200 \\ & & 11000 & & \end{array}$$

In subtraction, if you add or subtract the same value from the minuend and subtrahend it does not alter the answer.

$$\begin{array}{rcrcr} 28 & & (28 + 1) & & 29 \\ \underline{-\ 9} & \rightarrow & \underline{-(9 + 1)} & \rightarrow & \underline{-10} \\ & & & & 19 \end{array}$$

Thus, subtraction problems with certain types of numbers may be simplified through compensation without concern for adjusting the answer once it has been found.

$$\begin{array}{rcr} \textit{Old Problem} & & \textit{New Problem} \\ \$516.14 & \rightarrow & \$500.00 \\ \underline{-315.76} & & \underline{-300.00} \\ & & \$200.00 \end{array}$$

Perhaps the most important word to remember when teaching computation estimation is *experience*. Although many people develop their own mental computation techniques as adults, other people do not and go through life mathematically handicapped. In order to help people develop efficient and accurate mental computation techniques, we need to spend the necessary instructional and practice time on this important skill. Until textbook series include abundant activities involving estimation and mental arithmetic, teachers need to develop their own activities and make use of the resource materials available. Some of the resources are listed in the suggested readings.

SUGGESTED READINGS

Johnson, D. C. Teaching estimation and reasonableness of results. *Arithmetic Teacher* 27(1), 1979: 34–35.

O'Daffer, P. A case and techniques for estimation: Estimation experiences in elementary school mathematics—essential not extra. *Arithmetic Teacher* 27(6), February 1979: 46–51.

Reys, R. E., and Bestgen, B. J. Teaching and assessing computational estimation skills. *Elementary School Journal* 82(2), November 1981: 117–121.

Seymour, D. *Developing Skills in Estimation*. Palo Alto, CA: Dale Seymour Publications, 1981.

Trafton, P. Estimation and mental arithmetic: Important components and computation. In *Developing Computational Skills*, edited by M. N. Suydam and R. E. Reys (1978 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1978.

QUESTIONS

1. Select a recent elementary textbook series and discuss how estimation/mental computation is taught in the series. Is this topic integrated into the textbook or is it relegated to a separate chapter?

2. Describe the mental computation techniques that you use. Where did you learn these techniques? Do you think that these techniques should be taught in schools? Why or why not?

3. Question a number of working adults about the kinds of mental computation they do. When do they do mental arithmetic? Do they have any strategies that they have developed themselves?

4. Use the strategies suggested above to estimate the answers to the following Describe the strategies you have used.

a. $$\begin{array}{r} 3678 \\ 4927 \\ 7245 \\ \underline{+\,10357} \end{array}$$

b. $19\overline{)4283}$

c. $\frac{4}{9} \times 306$

d. $$\begin{array}{r} \$506.45 \\ \underline{-\,\$\ \ 97.45} \end{array}$$

e. 23% of $4000.00

CHAPTER

7

Logo in the Mathematics Classroom

Logo was the first language to be designed for children. Its developer, Seymour Papert, worked in Switzerland for several years with Jean Piaget before returning to the Massachusetts Institute of Technology where he developed a language that would allow children to explore mathematical (and other) ideas and to use the computer as a tool for expanding thinking processes. The language, Logo, allows children to build pieces of programs, then to use them as building blocks for developing more powerful programs. These, in turn, can be used to build even more complex programs, and so on. Fixed languages such as BASIC allow this sort of powerful program development to a much lesser degree, if at all.

Recent microcomputer versions of the language also include a child-oriented graphics language called "Turtle Graphics." Rather than using a fixed coordinate system for plotting points and lines, Turtle Graphics commands move a triangle (the turtle) around the screen ahead or backward and turn the turtle to the right or left depending on its present orientation and location on the screen. Papert's theory was to give children a graphics tool that was within their frame of reference—the movements of the turtle could be modeled by the movement of a robot turtle around the floor and by their own movements. After sufficient modeling experiences, they would be able to visualize the movements of themselves or the robot to help them write the appropriate Logo commands to draw the desired pictures on the screen.

Because of all the publicity Turtle Graphics have received, Logo is often thought of as primarily a graphics language. Unfortunately, many teachers confine their instruction to Turtle Graphics and related commands. Although it is true that Logo contains powerful graphics capabilities, it also includes a full range of text and number processing commands and capabilities. For example, it contains list processing commands that allow the programmer to create and manipulate lists of information.

This chapter contains a brief introduction to Logo. It covers the commands used throughout the rest of this book—those needed for programming activities designed to develop understanding of mathematics. You should be able to complete it in two to three hours, and, you should be able to cover the same commands in your classroom in several hours of instruction and hands-on activities, depending on the age of your pupils.

If Logo is unavailable to you, you can complete most of the suggested programming activities in this book in BASIC rather than Logo since many involve the use of the PRINT command which is similar in BASIC and Logo. Appendix A provides a tutorial in relevant BASIC commands. Note that in the following discussion Apple Logo commands are used. Logo commands in other versions of Logo (Terrapin Logo, IBM Logo, etc.) will be similar but you may find that commands for doing tasks such as clearing the workspace or saving programs are somewhat different. Consult the reference manual that accompanies your program for such details.

TURTLE COMMANDS

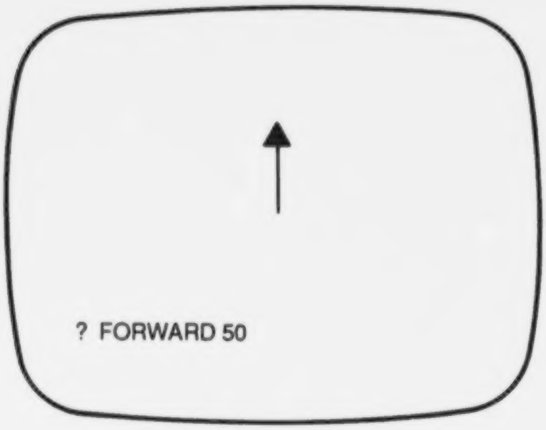
Since Logo is usually introduced through graphics in the *immediate mode*, we begin there. Once you have booted Logo, you may enter one or more Logo commands on a line then press RETURN and the computer will carry out that command immediately. This contrasts with the *procedure mode* where you may enter a whole sequence of commands then execute them at one time by typing the name of the procedure. The immediate mode is valuable when first introducing Logo graphics commands, when using the PRINT command to find the answer to an arithmetic operation, or when using Logo commands that perform a function like saving the work you have done or erasing the screen so that you may start a new design. After some introductory activities in the immediate mode, however, you should move children to the procedure mode since the immediate mode does not allow them to use the results they create while a program is running—the feature of computers that gives them their power and flexibility. For example, they would not be able to create a single sequence of commands to draw rectangles of different size, then use those commands 5 times to draw a truck. In the immediate mode, they must repeat the sequence of commands for drawing the rectangles each time they want to use them. Furthermore, if they create a design on the computer screen in the immediate mode, they have no way to go back and redo one or more of the commands to change their design. Once they have typed a command in the immediate mode, it is lost.

Before you can begin to draw shapes on the screen in the immediate mode, you must load Logo into the computer by booting the system with the Logo diskette. Next, type FORWARD 50 (or FD 50) to place the computer in graphics mode and to move the turtle 50 "turtle steps" ahead. You will notice the turtle (a triangle) at the top of a line on the screen. By typing FORWARD 50, we have made the turtle "walk" straight ahead while drawing a line.

You will notice that below the graphics area of the screen is a text window. The text window allows four lines of text to appear at the bottom of the screen while graphics are displayed above. This permits you to see the commands you type while they are being executed.

Next, make the turtle turn right. To do so, type RIGHT 45 (or RT 45) then press RETURN. Notice that when you turn the turtle, it only turns and does not go forward; type FORWARD 20 to get an idea of how far it turned. If you select a

FIGURE 7.1
The FORWARD statement.



```
? FORWARD 50
```

turn of less than 10, you do not see much of a turn. Experiment with FORWARD and RIGHT to see if you can make the turtle draw a square. Occasionally you may wish to send the turtle back to its "home" position (the center of the screen, pointing up) without clearing the screen. To do so, type the word HOME, followed by the RETURN key.

After you have done some experimenting, your screen will begin to get a bit messy. To put the turtle back in the home position *and* clear the screen, type CS (or CLEARSCREEN) followed by the RETURN key.

Note that you may place more than one command on a line if you separate them by one or more spaces.

```
CS FD 50 RT 90 FD 50 HOME <R>  (<R> stands for RETURN)
```

Two additional commands that work the same as FORWARD and RIGHT are BACK (BK) and LEFT (LT). Clear the screen, then experiment with these commands. See if you can draw a border around the edge of the screen.

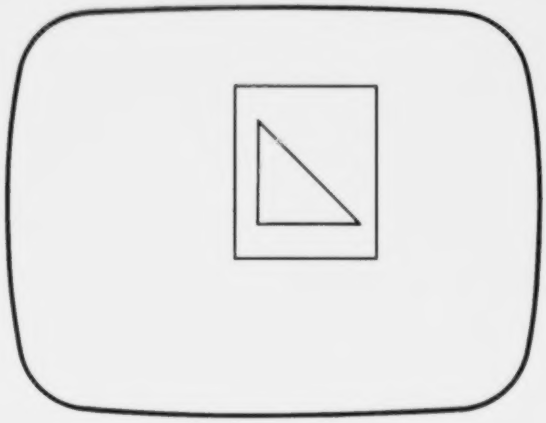
If you wish to draw a picture that has numerous parts, you can move the turtle without drawing. Imagine that the turtle has a pen that she can put down to draw on the canvas or pick up while moving so as not to draw. The command to pick up the pen is PENUP (PU) while the command to put the pen down in drawing position is PENDOWN (PD). The following sequence of commands draws a rectangle with a right triangle inside. (See Figure 7.2.)

Activities

1. Use your turtle to write your first name in capital letters.
2. Use your knowledge of the Pythagorean theorem (see Chapter 16) to draw a right triangle that has legs of length 40 and 50. See if you can draw the triangle so that the turtle ends exactly where it started.
3. Use the commands introduced above to draw a trapezoid. Repeat for a regular pentagon and a parallelogram.

FIGURE 7.2
Screen display of immediate commands with PU and PD.

```
HOME
FD 100
RT 90
FD 70
RT 90
FD 100
RT 90
FD 70
RT 110
PU
FD 30
LT 20
PD
FD 50
RT 135
FD 70.7
RT 135
FD 50
```



Two Editing Features in Logo

Different Logos have different editing commands. The following are Apple Logo editing commands, which may be similar to other Logos.

CTRL D or DELETE (Erases one character to the left of the cursor)
CTRL F (Erases the character the cursor is on)

The REPEAT Statement

The REPEAT statement allows us to condense greatly the number of statements needed to draw regular geometric shapes or perform repeated arithmetic operations. The form of the REPEAT statement and the resulting display are shown in Figure 7.3.

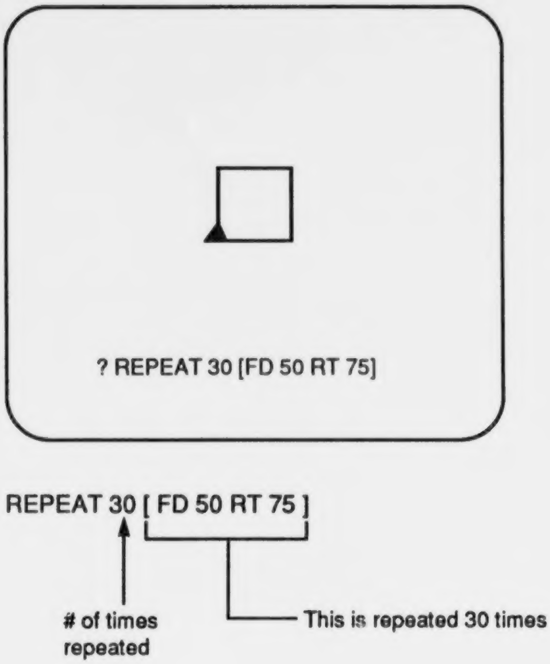
Whatever is between the brackets will be repeated the indicated number of times. On older Apple computers, type SHIFT N to get the left bracket, and SHIFT M to get the right bracket. Newer Apples and IBM-type computers have brackets on the keyboard. Figure 7.4, for example, shows a rectangle drawn with REPEAT.

```
REPEAT 2 [FD 60  RT 90  FD 100  RT 90]
```

You may put any legal Logo command inside a REPEAT statement. For example, if you want to draw six squares, each a slightly different angle from the other, use the following command:

```
REPEAT 6 [REPEAT 4  [BK 50  LT 90]  LT 20]
```

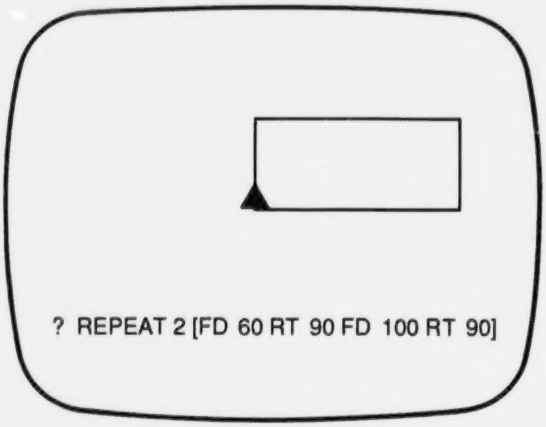
See Figure 7.5.



```
? REPEAT 30 [FD 50 RT 75]
```

```
REPEAT 30 [ FD 50 RT 75 ]
```

FIGURE 7.3
The REPEAT command.



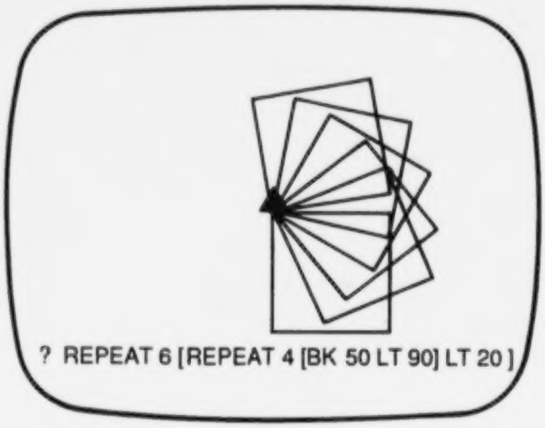
```
? REPEAT 2 [FD 60 RT 90 FD 100 RT 90]
```

FIGURE 7.4
Drawing a rectangle using REPEAT.

Questions

1. Use the REPEAT statement to draw a square, a regular pentagon, and an equilateral triangle.
2. See if you can use the REPEAT command to draw what looks like a circle. (Hint: a regular polygon with a large number of sides looks like a circle.)
3. Draw two concentric circles.
4. Use one REPEAT inside another to draw a "flower" made of long, rectangular petals.

FIGURE 7.5
Drawing a sequence of squares using REPEAT.



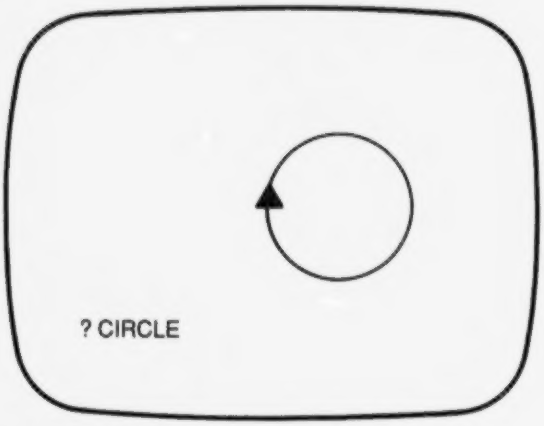
```
? REPEAT 6 [REPEAT 4 [BK 50 LT 90] LT 20 ]
```

PROCEDURES

A procedure is a sequence of statements that logically go together. For example, if you want to fit a rectangle, two squares and two circles together to draw a "car," you could create three procedures—one for the rectangle, one for a square, and one for a circle—and call them up after positioning the turtle at the appropriate places to make your drawing.

To create a procedure, type TO and the name you wish to give it. You use the name later when you execute the procedure. Note that as you type the statements in the procedure, they are not executed. They will be executed when you type the name of the procedure they are in. The following is a procedure for drawing a "circle." (See Figure 7.6.)

FIGURE 7.6
Drawing a circle.



```
? CIRCLE
```

```
TO CIRCLE
REPEAT 36 [FD 5 RT 10]
END
```

Be sure to hit RETURN after each line and to include an END statement at the end of the procedure.

Once the procedure has been entered, it may be *defined* by typing END when first entering the program or by pressing the Open Apple button and A simultaneously from the edit mode (see below). To run the procedure after it has been defined, simply type its name and hit RETURN. Type CIRCLE (and hit the RETURN button) to run the procedure "circle." If you discover that you have made a mistake in a procedure after it has been defined, type the words EDIT "CIRCLE" (or the name of any other procedure you wish to edit) and use the editing commands presented next.

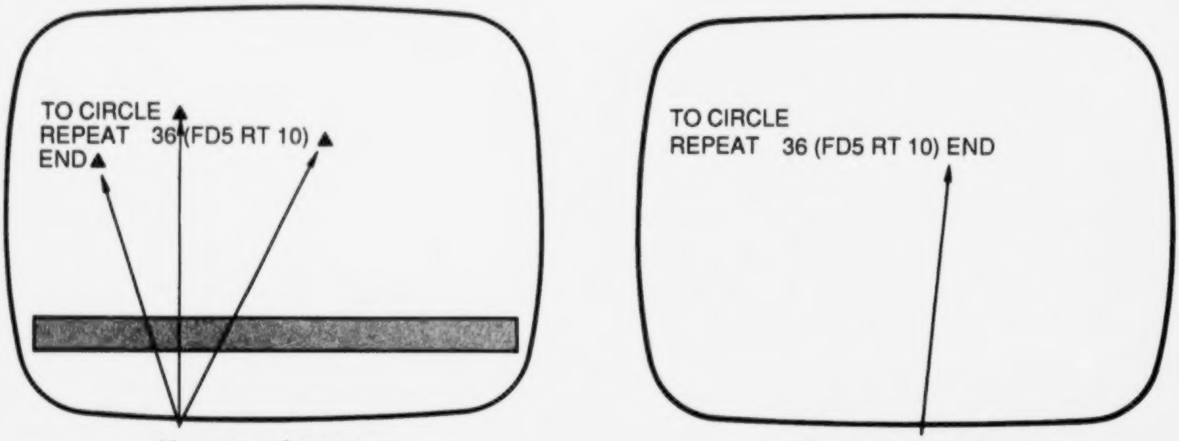
Editing Procedures

If you wish to make changes in lines previously entered, you may use the following editing commands to move the cursor to the desired location or to delete a line. Techniques for deleting characters are discussed above.

↑ (Moves the cursor up one line)
↓ (Moves the cursor down one line)
CTRL X (Deletes a line)
→ (Moves the cursor right without changing letters)
← (Moves the cursor left without changing letters)

To add characters in the middle of a line, simply position the cursor and type the characters. The text to the right of the cursor will move over as you type. Note that each line has an unseen "carriage return" (the name is a hold-over from mechanical typewriters) at the end that forces an end to the line and causes the next line to be printed on the following row. To verify that it is there, you may position the cursor immediately after the last character in a line (use the up and right arrows) and press CTRL F (or whatever command deletes the character the cursor is on in your version of Logo). The following line will jump up to the position where the cursor is. To reinsert the carriage return, hit the RETURN key and the line will return to its former position. If, when creating a procedure, you have lines that run together, you may have forgotten or deleted the carriage return. (See Figure 7.7.)

FIGURE 7.7
The role of the carriage return.



Unseen carriage returns

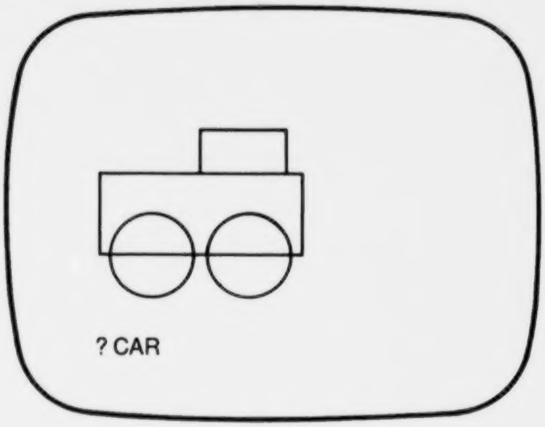
Carriage return deleted

Question

1. Write and run a procedure that draws an equilateral triangle.

Procedures Inside Other Procedures

One of the powerful features of Logo is the ability to call procedures from inside other procedures. For example, you might wish to draw a scene on the screen using geometric shapes. To do this you might create procedures to draw circles, rectangles, squares, triangles, etc., then move the turtle around the screen using PU and PD, then draw the shapes as desired to make your picture. The following procedure uses the procedures circle, rectangle, and square to draw a car. (See Figure 7.8.)



```
? CAR
```

FIGURE 7.8
The screen display for drawing a car.

```
TO CAR
  LT 90
  RECT1
  PU RT 90
  FD 40
  LT 90 FD 10
  PD RECT2
  LT 90
  PU FD 40 RT 90
  FD 25 RT 90 PD
  CIRCLE
  LT 90 PU
  FD 50 RT 90 PD
  CIRCLE
END
```

where the procedures used in CAR are as follows:

```
TO CIRCLE
  REPEAT 36 [FD 3 RT 10]
END
```

```
TO RECT1
  REPEAT 2 [FD 100 RT 90 FD 40 RT 90]
END

TO RECT2
  REPEAT 2 [FD 50 RT 90 FD 30 RT 90]
END
```

Now that you have a procedure named CAR, you could use it in a scene with a road, house, trees, etc. by including its name in the next procedure. It is essential, however, that all procedures used in another procedure have been defined.

Question

1. Use procedures that draw geometric shapes (rectangles, circles, etc.) to draw a scene with a car, road, house, and tree.

SWITCHING BETWEEN GRAPHICS AND TEXT

So far we have only worked with graphics commands. However, if we are asking the computer to solve and print out results of mathematical problems, we probably do not need a graphics display. When the computer is booted, it is automatically in the text mode (as it is in the edit mode). In Logo, whenever the computer encounters a graphics command, the top part of the screen is dedicated to graphics. To switch between these modes, use the following commands either in the immediate mode or in procedures:

TEXTSCREEN or CTRL-T (The entire screen can display text)
SPLITSCREEN or CTRL-S (Displays graphics with a 4-line text window at the bottom of the screen)
FULLSCREEN or CTRL-L (The entire screen displays graphics—no text window at the bottom)

THE PRINT COMMAND

The PRINT command is useful for printing text on the screen and for doing arithmetic operations. If you would like to have the computer print a sentence on the screen (e.g., you might want to print "THIS IS A RECTANGLE" under the drawing you just created), type

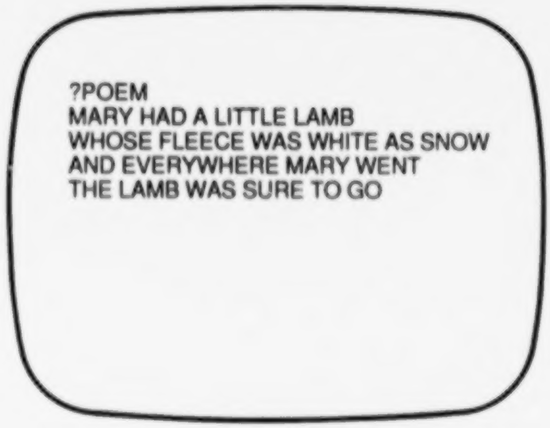
```
PRINT [THIS IS A RECTANGLE]
```

Whatever is between the brackets will be printed literally on the screen, including spaces and punctuation marks. This command, of course, may be used in procedures similarly to the graphics commands presented above.

```
TO POEM
  PRINT [MARY HAD A LITTLE LAMB]
  PRINT [WHOSE FLEECE WAS WHITE AS SNOW]
  PRINT [AND EVERYWHERE MARY WENT]
  PRINT [THE LAMB WAS SURE TO GO]
END
```

This procedure will print the poem on the screen when you type POEM. (See Figure 7.9.)

FIGURE 7.9
The screen display for POEM.



```
?POEM
MARY HAD A LITTLE LAMB
WHOSE FLEECE WAS WHITE AS SNOW
AND EVERYWHERE MARY WENT
THE LAMB WAS SURE TO GO
```

One use of the PRINT statements is for doing arithmetic. If you follow the PRINT statement with an arithmetic operation, the computer will carry out the operation before it prints the answer. For example,

```
PRINT 4 + 45.6
```

will print the result (49.6) on the screen. This command may be used in procedures or may be used in the immediate mode to find answers to operations that your calculator is not capable of handling.

With the exception of multiplication, the computer uses the standard symbols for doing operations on numbers: + for addition, − for subtraction, and / for division. For multiplication, it uses the symbol * rather than × . Thus, if you wanted to multiply 5 by 4 and subtract 15 from the result and print the result, you would type

```
PRINT 5 * 4 - 15
```

Logo, like other computer languages, uses the standard order of doing arithmetic operations—it does all multiplications and divisions first from left to right followed by all additions and subtractions from left to right. Thus if you enter

```
PRINT 5 + 4 * 6
```

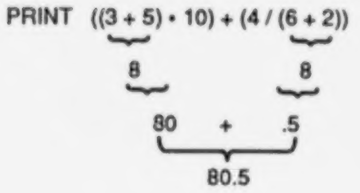
the computer multiplies the 4 by the 6, then adds 5 to the result, obtaining an answer of 29. To change this order of operations, use parentheses—the computer will do operations in parentheses first (from the innermost to the outermost if there are parentheses inside parentheses). Thus, if you enter

```
PRINT (5 + 4) * 6
```

the computer will print 54 as the output. The order of operations in the following PRINT statement is shown in Figure 7.10.

```
PRINT ((3 + 5) * 10) + (4 / (6 + 2))
```

FIGURE 7.10
The order of operations in Logo.



Chapter 11 discusses the order of operations more fully.

Questions

1. Use PRINT statements in a procedure to print a short letter to a friend.
2. Use the PRINT statement to calculate the following:
   a. The sales price of an item with a regular price of $49.95 but marked down 25%.
   b. The total price of items costing $125.45 and $321.98 including 5% sales tax.

VARIABLES

What makes a computer different from a calculator is its ability to do a computation, store the results, then use that result in another computation without any human invention. With a calculator, you could determine how much money you would have if you deposited $100 in an account that yielded 10% interest (compounded annually) for 20 years by calculating the interest for the first year, adding it to $100, calculating the interest for the second year, adding it to the amount in the account, and so on. With a computer, however, you can write a program to calculate the interest and add it to the total, then use this new amount to compute the new interest. Once you have written a program to determine the answer to this problem, the computer does all the rest of the work. With a calculator, the user must carry out each iteration.

What makes this type of computation possible is the **variable**. Variables can be thought of as slots in the computer where values are stored for later use. For example, we could tell the computer to store $100 in a memory location called M1 (for Money 1), then multiply the amount stored in M1 by .1 and add the new value to the amount stored there. If we repeated this process 20 times, we would have the amount in the savings account after 20 years.

To create a variable, we use the command MAKE.

```
MAKE "M1 100
```

This statement sets up a storage place called "M1" and stores the amount 100 in it:

M1
100

If we later want to use this variable in a PRINT statement or in a computation, we put a ":" in front of it to distinguish it from a word or Logo command. To print the contents of "M1", for example, we would type

```
PRINT :M1
```

To add something to a storage location already established, use statements similar to the following:

$$\texttt{MAKE "}\overset{\uparrow}{\texttt{M1}}\ \underbrace{\texttt{:M1 + .1 * :M1}}$$

Assign the results of
this operation to M1

This statement multiplies M1 by .1 (remember, the computer multiplies before it adds), adds the result to M1, then stores the new value into M1.

Putting this all together, a procedure to calculate the amount in a bank account if $100 is deposited and earns 10% interest for 20 years would be as follows:

```
TO BANK
  MAKE "M1 100
  REPEAT 20 [MAKE ``M1 :M1 + .1 * :M1]
  PRINT :M1
END
```

If you wanted the computer to print the results of each computation rather than just once at the end, you could put the PRINT statement inside the REPEAT statement:

```
TO BANK
  MAKE "M1 100
  REPEAT 20 [MAKE ``M1 :M1 + .1 * :M1 PRINT :M1]
END
```

Another way to create and use a variable is to include the variable name after the procedure name in the first line of the procedure.

```
TO SQUARE :S
```

When you type the name of the procedure to execute it, you include a number after the name. The variable and the number are stored in the computer's memory and can then be used later in the procedure.

The following procedure draws a square of variable size. It has a variable (:S) in the first line after the name of the procedure.

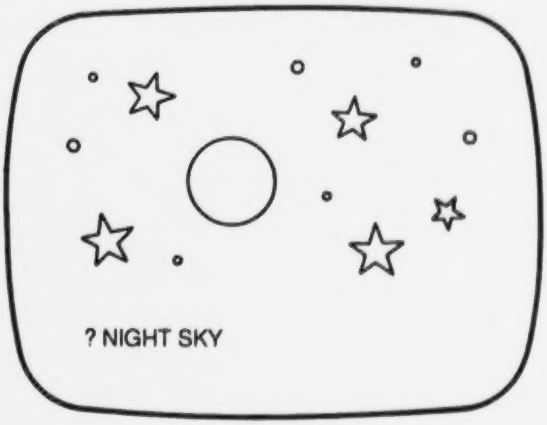
```
TO SQUARE :S
  REPEAT 4 [FD :S RT 90]
END
```

After the procedure has been defined, you type the procedure's name followed by a number (e.g., SQUARE 50) to call the procedure and assign the value to :S. This value is subsequently used in place of :S in the REPEAT statement and the procedure draws a square with that length for sides. Thus, one procedure makes it possible to draw squares any size you wish.

Questions

1. Write a procedure that uses a variable to print the sequence 1, 2, 4, 8, 16, 32, 64, 128, etc. (20 total numbers).
2. Write a procedure that uses the REPEAT command to make a five-pointed star of variable size and a second that makes a circle of any size. Use these two in a third procedure to draw the night sky with different-sized stars, the moon and several (small) planets. (See Figure 7.11.)

FIGURE 7.11
The screen display for a "night sky."



CHAPTER

8 Early Concepts

Children come to school with a wealth of background experiences and information related to mathematics. They have joined and separated sets (i.e., they have added and subtracted), they have classified objects according to various attributes, they have begun to use counting numbers, they have experienced terms such as *nothing, three,* and *less than,* to name a few. They are veritable storehouses of mathematical knowledge. Of course, the level of knowledge and understanding of children coming to school varies greatly, depending on their level of development and the type of guidance and experience they have had at home. It is important, even in kindergarten, to consider individual differences.

When children come to kindergarten, we expand and build upon their base of knowledge, starting with many of the same experiences they have had with their toys and other objects in their environment. We have them classify, sort, group, count, match, and determine relative location. Slowly, we begin to associate symbols with the manipulations. At the same time, we should avoid moving too quickly away from materials. Manipulations should be the foundation of mathematics throughout the elementary grades.

This chapter presents a discussion of the types of mathematical activities usually included in kindergarten programs. Many children, however, will not have mastered all of the following objectives before they arrive in the first grade. Indeed, since these skills form the foundation for mathematics in the later grades, you may find a need to return to these objectives with certain older children—even some in the middle and upper grades.

Behavioral Objectives for Pre- and Beginning Numbers

A. Classify and group objects according to a given attribute.
B. Identify the attribute by which objects are grouped.

C. State the relative location of two or more objectives.
D. State the relative size of two or more objects.
E. Use one-to-one matching to determine the relative size of two groups (equivalence, more than, less than).
F. Match two groups (where the number of objects in one group is a multiple of the number of objects in the other) one-to-many.
G. State the cardinality (number of objects) of a set with one, two, three, four, or five objects.
H. State the cardinality (number of objects) of an empty set (concept of zero).
I. State the cardinality of sets with six through twelve objects.
J. Write the numerals 0 through 9 when shown the corresponding set.
K. Make a set of objects to represent each of the numerals zero through 9.

NUMERATION SYSTEMS

Before beginning a discussion of methods for teaching early mathematical concepts, it is appropriate to provide some background about the system of numbers we use in our society. Where did it come from and what are its characteristics? The system of numeration that we use every day was not handed down from on high on a tablet of stone: it evolved. Examining this evolution of our system and of alternative numeration systems is useful because it not only helps us appreciate and understand our own system, but it helps us understand the many other systems that we use. Many of these other systems involve measurement or are related to the use of computers. It is interesting to note that the way we teach number concepts to children parallels, to some extent, the way the use of numbers has evolved in our civilization.

Early Number Concepts

The first method for keeping track of objects was through one-to-one matching. If a man wanted to keep track of his sheep, he would place one pebble in a container as each sheep left the compound to go to the pasture. As the sheep returned the pebbles were removed and if any were left after the last animal entered, sheep were missing.

This type of numeration system is often used, even today. For example, a teacher might use tally marks to keep track of the number of test grades in the A, B, C, etc., categories after they have been graded. This information is then shared with the class so that students can evaluate their performances and compare them to those of the rest of the class.

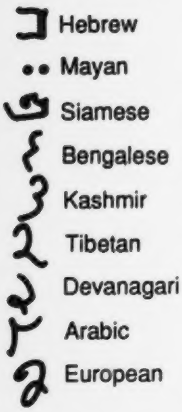
When people needed to keep track of larger groups of objects, one-to-many matching techniques were used—a marker (such as a stick or tally mark) would stand for a group of objects rather than a single object. Since people always carry an excellent counting device—fingers—with them, it was natural to group by fives or tens. Early records show a variety of grouping techniques, but the most common was grouping by tens.

A major step in the development of more sophisticated numeration procedures was the use of verbal names—and eventually written symbols—to represent numbers. Thus, a set of six objects could be retained mentally with a single word rather than a physical collection of objects or a mental picture of the objects. The power of this development and its impact on society were probably just as great as the invention of the wheel or the computer.

The names for numbers provided a powerful tool that allowed a person to keep, determine, and compare quantities quickly and easily. This tool was counting, the basis for both ordinal and cardinal numbers. *Cardinal numbers* tell us how many while *ordinal numbers* refer to the relative position of an object in a sequence. The cardinal number *five* is the entire set of five objects, while the ordinal number *fifth* refers to a specific object counted. In a sense, ordinal counting is one-to-one matching and cardinal counting (counting to find "how many") is one-to-many matching.

The distinction between ordinal and cardinal number is important in teaching number concepts to young children because they need to realize that "eight marbles" means the entire set of eight marbles and not just the eighth one counted. Instructional techniques regarding these concepts are discussed later in this chapter.

FIGURE 8.1
Different names for "two."



Symbols for Numbers

Early civilizations had a variety of written symbols for the smaller numbers. Figure 8.1 contains several of the names that have been used for two.

The symbols we use to represent numbers are part of the Hindu-Arabic system of numeration. The numerals 1 through 9 can be traced to a number of different ancient civilizations, but the discovery of the concept of zero (or at least written reference to it) did not occur until the ninth century A.D. The first written reference to the elements of our numeration system including the concepts of zero and place value were made by a Hindu monk in about 825. This system came to the attention of Arabic scholars and was refined and spread around the world.

ACTIVITIES FOR TEACHING PRE- AND BEGINNING NUMBERS

The first step in building an understanding of the concept of *number* and the individual number concepts (*one, six,* and so on) is to develop the idea of a set or group. Early activities with grouping should be done without any reference to the number of objects in the group. Note that while the foundation for the number concepts is being built, other important concepts and vocabulary are being developed.

Activities for Grouping Concepts (Objectives A–B)

One way to develop the concepts of *grouping* and *classifying* is to have children sort familiar objects. Involve the students in many and varied sorting activities, classifying and grouping these objects.

- Set up a "Math Fun Box" containing a wide variety of small objects that can be used in classifying activities. The following are some examples of materials that could be included.

Short pencils
Long pencils
Pencils of different colors
Pens of different colors
Large and small paper clips
Different-colored gum erasers
Small plastic toys
Plastic cups
Small, smooth rocks
Poker chips
Colored rods
Popsicle sticks
Knives, forks, spoons
Small, plastic bowls
A pair of scissors
A piece of chalk
Coins
Combs of different colors
Different small pails
Match boxes, pill boxes

- Direct various classification activities. Ask the pupils to make a group of objects that are yellow. Ask them to make a group of things that are pointed. Ask them to classify objects according to function (e.g., all objects from which you could drink water).

- Set up a card file with pictures to indicate the type of classification to the student. Place the card file next to the Math Fun Box and allow certain times for a student to go back to the box and do the activities. Figure 8.2 shows examples of the activity cards. Obviously, students need some direction the first few times they do this type of activity.

Cut a set of *attribute pieces* from tagboard for each group of three children in your class. The set should contain 32 pieces with each piece in one of four colors, two sizes, and four shapes. To make your set, copy the shapes in Figure 8.3 on red construction paper or tagboard, then on blue, yellow, and green, and cut them out.

- Have the student group the pieces according to single attributes—*small* or *yellow*, for example. Discuss the attributes of the groups they make.

- Start to make a set by placing one object in a container such as an open shoebox. Ask the student to place another shape that is the same. Ask why she chose that particular piece. Ask her to place more of the shapes in the container, using the same attribute. Discuss as you continue the activity.

- Have the student make a set with two or three pieces without telling you the attribute she is using for classification. Select pieces from the unused pile of pieces and ask her to indicate whether they belong or not.

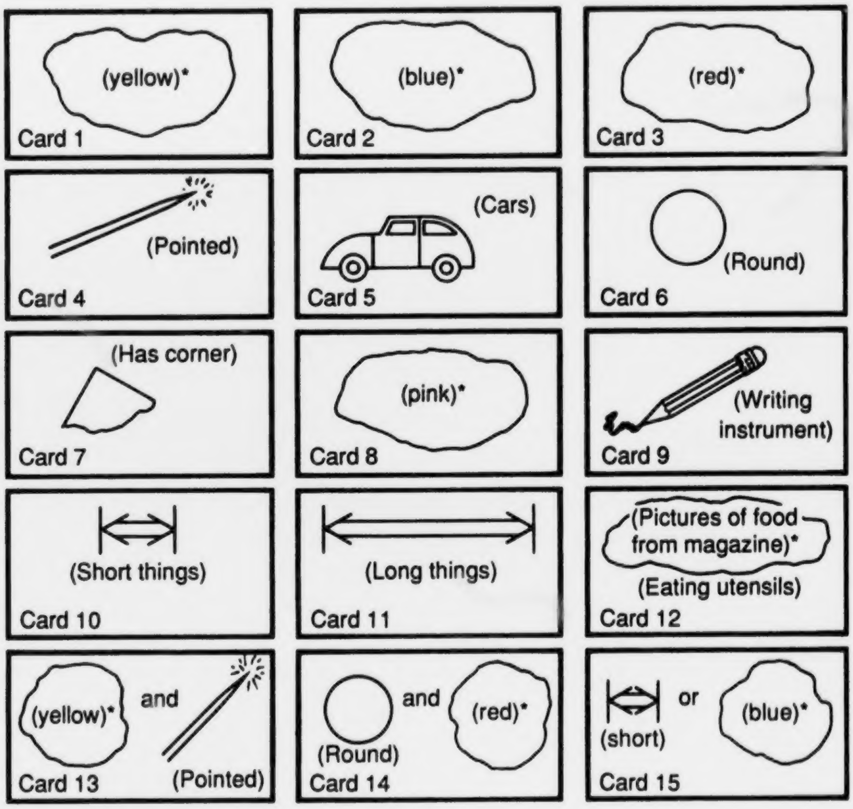
- Show the student how to classify objects according to two attributes, for example, small and yellow. Carry out activities similar to those above but using two attributes.

- Do activities in classification with pupils in the classroom. For example, ask all children with black hair to go to a certain corner of the room.

Activities for Relative Location and Size (Objectives C–D)

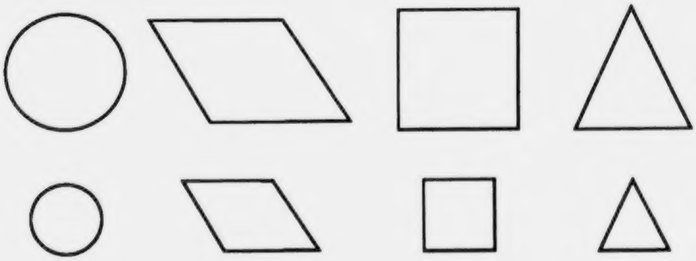
As in grouping, experiences with these concepts can involve objects from in and around the classroom. Children should manipulate, explore, and play games to develop and reinforce concepts of relative location and size.

FIGURE 8.2
Sample activity cards for a math fun box.


(yellow)*
Card 1
(blue)*
Card 2
(red)*
Card 3
(Pointed)
Card 4
(Cars)
Card 5
(Round)
Card 6
(Has corner)
Card 7
(pink)*
Card 8
(Writing instrument)
Card 9
(Short things)
Card 10
(Long things)
Card 11
(Pictures of food from magazine)*
(Eating utensils)
Card 12
(yellow)*
and
(Pointed)
Card 13
and
(red)*
(Round)
Card 14
(short)
or
(blue)*
Card 15

* Do not print these words; instead put the actual color or picture on the card.

FIGURE 8.3
Attribute pieces.



- Demonstrate with books, boxes, toys, and other objects the meaning of *above, below, between, middle, under,* and *over.* Ask the pupil to use similar materials to demonstrate these words. For example, "Johnny, hold the book below the table."

- If working with several pupils, use them to demonstrate terms like *right, left, between,* and *in the middle.* Give them directions to follow such as, "Renee, please stand to the right of Jesse."

- Place objects in desired locations and ask the students to state where one is in relation to another. Make a stack of four books and ask, "Rodney, where is the red book?"

- Draw pictures on paper or on the blackboard and discuss their relative location.

- Carry out similar activities for the words *larger, smaller, largest, smallest,* and *the same size as.*

One-to-One and One-to-Many Matching (Objectives E–F)

- Obtain sets of objects that naturally go together—flowers and vases, cups and saucers, or paper dolls and dresses, then demonstrate how to match them one-to-one. Do activities where each set has the same number of objects and others where one set has more than another.

- Demonstrate one-to-one matching of sets that are not naturally associated with each other, like popsicle sticks and dried beans. Use this method to determine the relative size of pairs of sets. Discuss why one set has *more, less,* or *the same number as* the second set.

- Have the student place a popsicle stick in a cup or box for each object in a set containing between ten and twenty objects (e.g., desks in the room).

- Have the student draw tally marks for each object in a set of between ten and twenty objects.

- Do activities similar to those above, but where the number of objects in one set is a multiple of the number of objects in the other set. Flowers and vases could be used, with two flowers for every vase. Do not tell the student how many flowers go in each vase; instead say "Put flowers in the vases so that each vase has the same number of flowers."

- Do similar activities with pens and pockets, popsicle sticks and cups, and so on.

- Have the student place popsicle sticks in a given location to represent groups of a given number of objects. For example, "Put one stick in a box for each pair of students in a classroom."

- Ask the student to make a tally mark for each group of a given number of objects. Give the student 24 pinto beans and ask her to make a mark every time she counts three of them. Vary the type and number of objects.

Activities for the Numbers Zero through Twelve (Objectives G–K)

An instructional sequence for the beginning number ideas should start at *one* or *two.* Certain children may experience difficulties with the concept of one because we usually think of sets as containing more than one object and because of the variety of terms we use for the concept one (e.g., *a, the, it*). If this happens, work with *two* through *five*, then move back to *one* and *zero*.

The idea of zero is more difficult than the ideas one through twelve—symbols for this number were developed thousands of years after symbols for other numbers. The best method for approaching this concept is to teach the larger numbers first and then to work back down to zero, using materials like an empty loop of yarn or a shoebox.

Since no place-value or grouping is suggested by the verbal names *ten, eleven,* and *twelve,* beginning activities can be similar to those for *one* through *nine.* In the next chapter we will discuss ways to develop an understanding of place-value and of the written numeral for these numbers.

- Give each child five counting chips or popsicle sticks and a loop of yarn. Have them spread the loop on their desks. You should do the same on your desk so that all can see. Take one popsicle stick and ask the children to do the same. Place it in the loop. Discuss the number property of the set. For example, say, "This is a group of one. This is one popsicle stick." Point to one of the pupil's loops and ask her what it is.

- Repeat with a large variety of materials and grouping methods. Ask the pupils to make a group of one pencil, one book, one pupil, to hold up one finger, to touch one person, to point to one ear, or to stand on one foot.

- Start with a set of one object. Add another like object to the set and discuss its number property, as was done with the set of one.

- Do activities for *two* with a large variety of materials and grouping methods.

- Continue this same pattern for *three, four,* and *five.* Always start with a known group and work up. When *one* through *five* have been partially mastered, work with these numbers together. Make groups and ask the children to state the number property of the group. Ask Johnny, Misty, Juan, and Renee to stand in a large loop of yarn placed on the floor. Ask Steven how many pupils are in the set. Ask Renee and Misty to go back to their seats, then ask Misty how many are in the loop of yarn.

- Start with sets of five chips or popsicle sticks. Remove an object and discuss the number property. Repeat until there are no objects left in the set. Explain that this is a set with zero chips or popsicle sticks.

- Make other sets with the number property *zero.* Ask the children to touch zero people, hold up zero fingers, place zero pink elephants in their loop of yarn, point to zero rabbits, and so on. Be sure to use the term *zero* rather than *no* or *none,* when doing these activities.

- Start with a set of five and add an object. Discuss the number property of this set, as was done with the smaller numbers, and carry out further activities to reinforce the concept. Do similar activities with the numbers *seven* through *twelve.*

- Make number cards for each of the concepts *zero* through *twelve* on 21 by 28 cm. (8 ½ by 11 inch) laminated tagboard. Each card should contain a picture of a set and include the written numeral and number name. See Figure 8.4 for an example. You may wish to avoid using or discussing the *numerals* 10, 11, and 12 until work with place value has begun since the verbal names for these numbers are learned long before the written numerals.

  Place these cards so that they are visible and easily accessible to the pupils and allow them to use the cards to determine the numerousness of sets. This may be done by placing objects in sets on top of the pictures on the cards. If they match one-to-one they have found the number property of the set.

- After much experience with sets of objects and their number properties, allow the children to count to determine "how many." Use a variety of materials and

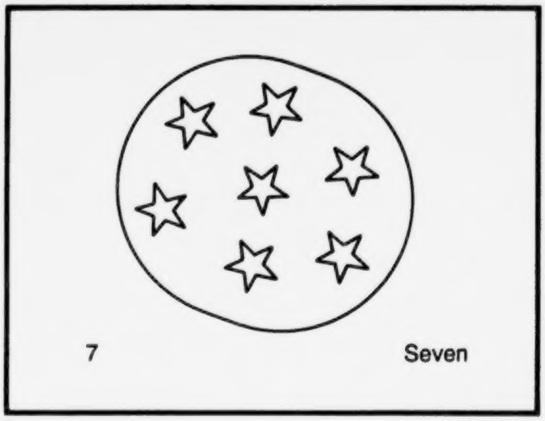


FIGURE 8.4
A number card example.

grouping techniques to make sets with zero through twelve objects and have the children count to determine the number properties of these groups.

- Associate written numerals with the number ideas *zero* through *nine*. Make a set of numeral cards on tagboard or posterboard so that the numerals are about 10 cm. (4 in.) high. Make a set with a given numerousness, hold up the appropriate numeral card, and discuss the meaning of the numeral in relation to the set. Do many activities like this with each of the numerals 0 through 9.

- Hold up a numeral card and ask students to make a set with "that many" objects in it. Make a set with zero through nine objects and ask the pupils to identify the appropriate numeral card.

- Make worksheets with pictures of objects and have the pupils write the appropriate numeral for each set. Include activities where students draw sets to represent the numerals 0 through 9.

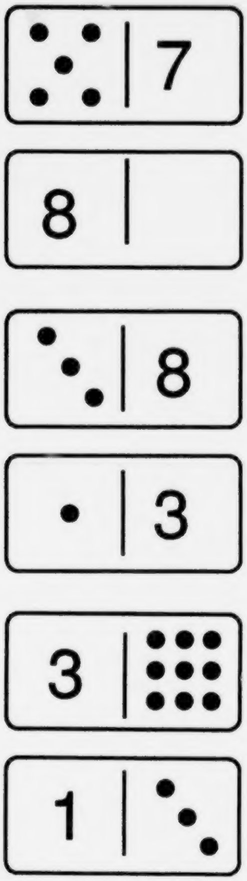
- Set up the class' calculators so that the constant has been programmed (see Chapter 3) to add 1s. Have the students count and push the = button each time they count a new number, watching the display so that they can associate the written (calculator) numeral with the spoken numeral. To start with, have them count only to 9. Later, count beyond 20.

- Program the constant function to count by 2s. Have the class or small groups of children count by 2s, starting at 2 or 1, and at larger numbers. (Of course, you need to properly set the calculator to correspond to the starting point of your counting.) Repeat, using other counting sequences—count by 3s, 4s, 5s, and 10s—and starting at various points.

- Have the children in your class write Logo procedures to draw a given set of objects between 0 and 9 on the screen. For example, ask them to draw 5 marks on the screen. Next, have them write a procedure (see Chapter 7) to draw the appropriate numeral next to the shapes.

SAMPLE DRILL AND PRACTICE ACTIVITIES

FIGURE 8.5
Number dominoes.



Activity 8.1: Number Dominoes

Objective: Associate a written numeral with a set of dots.

Materials: Twenty-one or more dominoes made from tagboard, with one side of the domino containing a numeral (0–9) and the other side containing a set of dots. (See Figure 8.5.) Standard dominoes may be modified for this game by taping a written numeral over the dots on one side of each domino.

Methods of Play:

1. The dominoes are placed face down in the center of the playing area and shuffled.
2. Each player draws one to see who starts—the person with the largest number (dots or numeral) starts.
3. The dominoes are replaced and reshuffled and each player draws 7 dominoes.
4. The person starting places a domino on the table. The next player may play a domino to match either end of the first one played—either picture to picture, numeral to numeral, or numeral to picture.
5. A "double" (same number represented on both sides of the domino) is played crosswise, allowing further play at either end. Note that a domino may be played only at the unmatched end of another domino.

Activity 8.2 Number Concentration

Objective: Associate a written numeral with a picture of a set.

Materials: Six pairs of 6 cm. by 6 cm. (2½ in.) laminated cards. One card of each pair should include a picture of a number and the other should contain the associated numeral. Include a pair of cards for zero.

Method:

1. Mix the cards and place them face down in a 3 card by 4 card array.
2. When it is a student's turn he turns up two cards. If they make a pair he keeps them, if not he returns them to their original position.
3. When the cards are gone the game is over and the person with the most cards is the winner.

Variation: Increase or decrease the number of pairs of cards depending on the ability of the players.

Activity 8.3 Number Puzzle

Objective: Associate a written numeral with a set of dots.

Materials:

1. Pencil and ruler or other straightedge
2. Worksheet, Figure 8.6

Method: Student uses a pencil and straightedge to connect the numeral with the corresponding set of dots. If done correctly the picture appears.

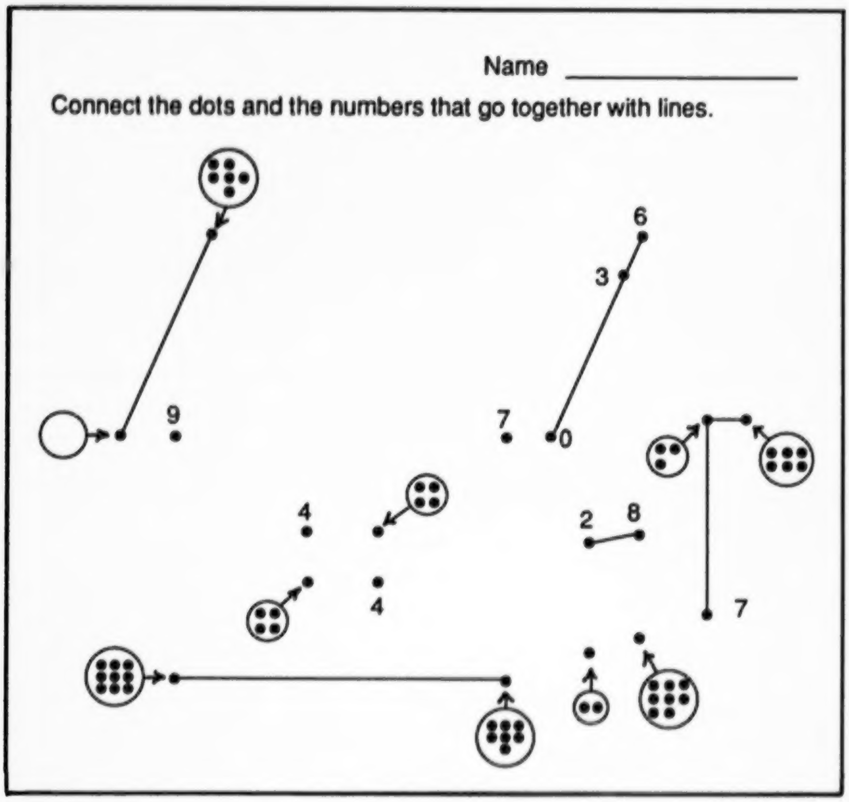


FIGURE 8.6
A number puzzle.

Name ______________

Connect the dots and the numbers that go together with lines.

SUGGESTED READINGS

Belk, E. C. Hop-a-long math. *Arithmetic Teacher* 31(8), April 1984: 26.

Bruni, J. V., and Silverman, H. Making and using attribute materials, *Arithmetic Teacher* 22(2), February 1975: 88–95.

Burton, G. M. Helping parents help their preschool children. *Arithmetic Teacher* 25(8), May 1978: 12–14.

Cruikshank, D. E., Fitzgerald, D. L., and Jensen, L. R. *Young Children Learning Mathematics.* Boston, MA: Allyn and Bacon, 1980.

Engelhardt, J. M., Ashlock, R. B., and Wiebe, J. H. *Helping Children Understand and Use Numerals.* Boston, MA: Allyn and Bacon, 1984.

Hollingsworth, C. Humanumerals. *Arithmetic Teachers* 33(2), October 1985: 12-14.

Horak, V. M., and Horak, W. J. Developing mathematical understandings with bead strings. *Arithmetic Teacher* 30(4), December 1982: 6-9.

Lettieri, F. M. Meet the Zorkies: A new attribute material. *Arithmetic Teacher* 26(1), September 1978: 36-39.

Liedtke, W. W. Rational counting. *Arithmetic Teacher* 26(2), October 1978: 20-26.

______. Young children—small numbers: Making numbers come alive. *Arithmetic Teacher* 31(1), September 1983: 34-37.

Lindquist, M. M., and Dana, M. E. Make counting really count—counting projects for first and second grades. *Arithmetic Teacher* 25(8), May 1978: 4-11.

Mathematics Learning in Early Childhood (1975 Yearbook of the National Council of Teachers of Mathematics). Reston, VA. National Council of Teachers of Mathematics, 1975.

Yvon, B. R., and Dopheide, J. D. Iggies come to kindergarten. *Arithmetic Teacher* 31(5), January 1984: 36-38.

Yvon, B. R., and Spooner, E. B. Variations in kindergarten mathematics programs and what a teacher can do about it. *Arithmetic Teacher* 29(5), January 1982: 46-52.

QUESTIONS

1. Gather a set of about 30 small objects like pinto beans. Ask a four- or five-year-old preschooler to count as many beans as he can. When he has counted the last number correctly, ask him to show you that many beans. What are the results? What does this tell you about the child's concept of number?
2. List ten different pairs of everyday materials that naturally go together for one-to-one matching activities.
3. Write a Logo program designed to be used by children to draw "pens" around a given number of animals (up to 14). First, develop a procedure that creates a shape resembling some animal. Write another procedure that uses this procedure to place 14 of the animals on the screen, with at least two centimeters between each. To use the program, ask children to make the turtle draw a "pen" around a given number (e.g., five) of the animals so that they cannot escape. Try it out with a child.
4. Write a lesson plan for adjectives of relative position. Start with concrete activities and include pencil and paper activities and a game.

CHAPTER

9 Multidigit Numerals

NUMERATION

Imagine trying to get along in our modern society without understanding numbers. When the cashier said, "That'll be twelve dollars and seventy-three cents, please," you would be lost. You would not be able to use a handheld calculator. You would have trouble communicating the correct amount of carpet needed to cover your living room floor. You would have a hard time finding a radio station on your stereo FM receiver or finding someone's house if you had never been there before.

Many elementary children, even in the upper grades, do not understand our system of numbers. They might be able to "read" certain numbers, select the larger of two numbers, and even do arithmetical computations with some success; but their lack of understanding of numbers makes them as functionally incompetent in arithmetic as a person who can pronounce written words but cannot understand them in prose. A knowledge of multidigit numerals is fundamental to success in elementary mathematics. If you look at the average elementary math textbook, you will see many pages of problems and activities involving whole numbers and, beginning with the first grade, most of these are multidigit. In a study by the author it was found that in a typical fourth-grade mathematics textbook, 90% of all student exercises involve multidigit numerals.

Not only are multidigit numerals used in most elementary computations, but many of our computational techniques (algorithms) are based on the structure of our numeration system. Because of the structure of our numbers, we can do many things that would be difficult using other systems. For example, imagine trying to do the following multiplication problem:

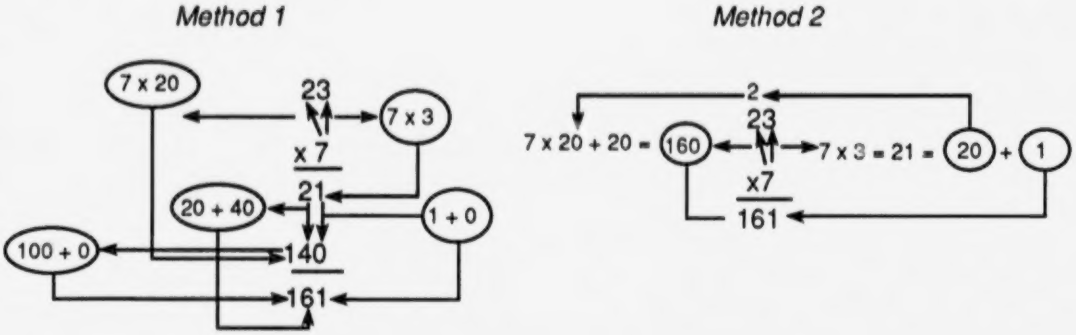
$$\begin{array}{r} \text{LXIV} \\ \times\ \text{XCII} \\ \hline \end{array}$$

As further verification that our computational techniques are based on the structure of our numbers, observe the following sequence for teaching meaning for multidigit multiplication taken from another elementary mathematics textbook:

$$
\begin{aligned}
23 \times 7 &= (20 + 3) \times 7 \\
&= (20 \times 7) + (3 \times 7) \\
&= 140 + 21 \\
&= (100 + 40) + (20 + 1) \\
&= 100 + (40 + 20) + 1 \\
&= 100 + 60 + 1 \\
&= 161
\end{aligned}
$$

This sequence involves breaking multidigit numerals down into smaller, more meaningful parts (i.e., writing the numerals in expanded notation). Our usual way of multiplying makes use of these same procedures and meanings, only in shortened form. (See Figure 9.1.) Therefore, in order to understand multiplication of multidigit numerals, the student must understand the meaning of multidigit numerals.

FIGURE 9.1
The steps of multiplication.



An understanding of multidigit numbers is also necessary for division. The first step in dividing 625 by 25 is to estimate the quotient of 620 and 25. Without an understanding of the meaning of the numerals 25 and 625, the chances of finding a correct solution to this problem are meager and the chances of using it in a real-life situation are even smaller.

Because the concepts involved in multidigit numerals are so complex and because most children will not understand purely abstract discussions, a sequence of physical models should be used in the beginning stages of instruction, leading up to a model that is similar to written numerals. Start with single objects like those used in teaching the single digits that can be grouped to show tens and hundreds. Later, use a physical model where place determines value, like our written numerals. Finally, much work should be done with written numerals themselves. After the instruction is finished, students should be able to name the place value, face value, and total value of every digit in the multidigit numerals; should be able to read numerals; should be able to write them in expanded notation; should be able to regroup, round, and determine which of several numbers is larger; and should be able to continue number patterns.

Behavioral Objectives for Numeration

A. Name the number of tens, ones, and (later) hundreds when given a model for a number.
B. Make a model for a number when told the number of tens and ones it has.
C. Name the number represented by a physical model for a number.
D. Make a model for a number when hearing the verbal name for the number.
E. Write the appropriate numeral when shown a model where place determines value (e.g., place-value chart).
F. Make a model for a written numeral using a place-value model.
G. Regroup with a physical model.
H. Represent multidigit numerals pictorially.
I. Interpret pictorial models for multidigit numbers.
J. Regroup with a pictorial model.
K. Read numerals.
L. Write verbally-stated numerals.
M. Name the number of ones, tens, hundreds, etc., when shown a written numeral.
N. Write a numeral that has a given number of ones, tens, hundreds, thousands, etc.
O. Identify the place value of given digits in written numerals.
P. Name the total value in a given place of a multidigit numeral.
Q. Write a given numeral in expanded notation.
R. Convert an expanded notation number sentence to a numeral.
S. Select the largest or smallest number when given two or more numbers.
T. Write the next in an ascending or descending arithmetic sequence of numbers.
U. Regroup from a given place to an adjacent place in a multidigit numeral.
V. Round a whole number to a given place.
W. Read and interpret large numbers (> 10,000).

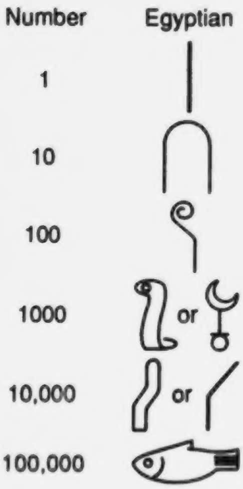
OTHER SYSTEMS OF NUMERATION

The previous chapter discusses the evolution of ways of keeping track of quantities, from the use of a stone or tally mark for each object to the use of number names and written symbols to represent how many there are in a group of objects. The next step in the sophisticated and more powerful use of numbers came when larger and larger quantities needed to be represented. Inventing and remembering a different symbol for each possible quantity would clearly be impossible, so many cultures used *additive* systems.

Egyptian Number System

An additive system uses a limited number of symbols to represent larger quantities, repeating the appropriate symbols as often as necessary in order to produce the de-

FIGURE 9.2
Egyptian number symbols.



sired value. For example, the earliest Egyptian number system used the symbols shown in Figure 9.2. The number 205,341 in this system would be as shown in Figure 9.3.

The value of the number is found by adding the values of all the symbols.

Note that the Egyptian numeration system does not make use of *place value*. Although the ancient Egyptians wrote the symbols for the largest values on the left and moved to the right for symbols representing smaller values, it would be possible to represent 205,341 as in Figure 9.4 without any loss of meaning—the relative position of a symbol in a number did not determine its value.

FIGURE 9.3
The Egyptian number symbols for 205,341.

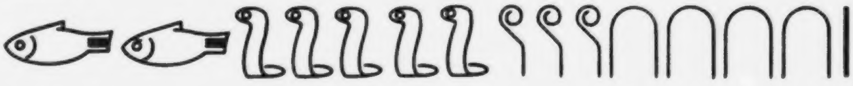
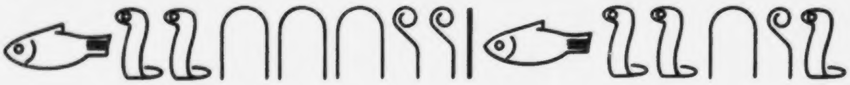


FIGURE 9.4
Another representation of 205,341 using Egyptian number symbols.



Roman Numerals

An ancient system still in limited use today—at least by the movie industry—is the Roman system of numeration. This system is additive, like the Egyptian system, but uses groupings by fives as well as by tens. Neither Roman numerals nor the Egyptian system were used in computation because the systems did not have a symbol for zero nor were they based on place value. The Romans used the abacus for their computations then recorded their results using their numerals.

The symbols used in the Roman system are shown in Table 9.1.

TABLE 9.1

Symbol	Value	Grouping		
I	1			
V	5	V = IIIII	or	5 = 5 × 1
X	10	X = VV	or	10 = 2 × 5
L	50	L = XXXXX	or	50 = 5 × 10
C	100	C = LL	or	100 = 2 × 50
D	500	D = CCCCC	or	500 = 5 × 100
M	1000	M = DD	or	1000 = 2 × 500

Other rules operate in the Roman system besides additivity. To represent the numbers 4, 9, 40, 90, 400, 900, and so forth, the symbol representing a smaller value is placed in front of a symbol of greater value (e.g., IV = 5 − 1; IX = 10 − 1; XL = 50 − 10; etc.). To represent values larger than 1000, a bar is placed over the symbol to indicate that you multiply the value of the symbol by 1000. The symbols $\overline{\text{X}}$ and $\overline{\text{D}}$ represent 10,000 and 500,000 respectively. Finally, since the order of two adjacent symbols may indicate subtraction, values are arranged from left to right in descending order. Note that Roman numerals do make limited use of place value—IX means 9 while XI means 11.

To determine the value of a Roman numeral, simply add up the values of all the symbols. If a smaller value precedes a larger value, subtract the smaller value from the larger, then add this value to the total. For example:

CCXXXIII = 200 + 30 + 3 = 233
MDCCLVIII = 1000 + 500 + 200 + 50 + 5 + 3 = 1758
MCMLXXXVII = 1000 + 900 + 50 + 30 + 5 + 2 = 1987
CXXMMMD = 100,000 + 20,000 + 3000 + 500 = 123,500

THE BASE-TEN, PLACE-VALUE SYSTEM

The numeration system that we use—the Hindu-Arabic system—has a number of important attributes:

Base Ten: A base-ten numeration system uses 10 symbols to represent a number of any size. Our system uses the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Place Value: As is mentioned above, the Hindu-Arabic numeration system is a place-value system, one where the value of a digit in a number is determined partly by its relative position in the number. In the number 523, for example, the place value of 5 is one hundred while the place values of the 2 and 3 are ten and one, respectively. If we change the place value of the 5 in 523 by adding a place-holding zero between the 5 and 2 (giving us 5023), the value of the 5 changes.

In a sense, the 5 in 523 has two meanings. Besides its place value it has a *face* value—the value of the digit in isolation. To find the actual value of 5 in the context of the multidigit number, we multiply its face value times its place value.

$$5 \times 100 = 500$$

To find the total value of the numeral 523, we need to calculate the total value of each digit and add them together.

$$523 = 5 \times 100 + 2 \times 10 + 3 \times 1$$

Since our system is base ten, each place is 10 times the value of its neighbor to the right. Thus we have a system based on powers of 10 (see Figure 9.5).

FIGURE 9.5
The powers of 10.

$10^0 = 1$
$10^1 = 10$
$10^2 = 100$
$10^3 = 1000$
$10^4 = 10{,}000$
$10^5 = 100{,}000$
$10^6 = 1{,}000{,}000$
and so on.

When we write a numeral as the *sum* of each face value *times* its place value as we did with 523 above, we have *expanded* the number. Writing a number in expanded notation helps make clear its meaning and is an important instructional tool. A number like 4518 can be expanded several ways:

4518 = 4 thousands, 5 hundreds, 1 ten, 8 ones
$4518 = 4000 + 500 + 10 + 8$
$4518 = 4 \times 1000 + 5 \times 100 + 1 \times 10 + 8 \times 1$
$4518 = 4 \times 10^3 + 5 \times 10^2 + 1 \times 10^1 + 8 \times 10^0$

In addition, we could state that 4518 has 3 thousands, 15 hundreds, 1 ten, and 8 ones, or that it has 4 thousands, 3 hundreds, 21 tens, and 8 ones, or even that it has 4518 ones! The process of representing a given quantity with groupings other than the standard (i.e., no more than 9 fits into any place) is called *regrouping*. Regrouping is essential to operating (adding, subtracting, etc.) on multidigit numbers.

Periods

Reading a number like 3470054124 is difficult because we read numbers from left to right and most people cannot tell at a glance what place the 3 is in. For that reason, we divide larger numbers up into periods. *Periods* are groups of 3 places in numbers having more than 4 places, separated by commas or spaces. Most English-language mathematics books separate the periods with commas.

3,470,054,124

There is a growing trend to use spaces instead of commas in large numbers, especially in our technologically-oriented society where computerized typesetters and electronic word processors can print half-spaces.

3 470 054 124

Another reason that spaces are replacing commas in large numbers is that typing commas in numbers having more than 4 places will result in error messages if you include them in numbers you enter into a computer.

OTHER BASE NUMBER SYSTEMS

Although the base-ten system is virtually universal for everyday number activities, there is nothing sacred about it. Indeed, if humankind had evolved with 8 fingers instead of 10, we would probably be using a base-eight system of numeration.

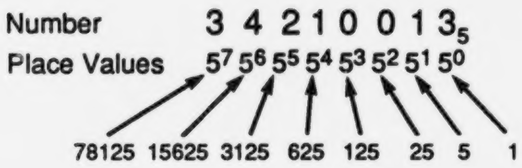
Other-based number systems have all the same characteristics as our base-ten system except that their place values are powers of numbers other than ten.

1. Each has a base that is used to determine the place values.
2. Each has a set of symbols—as many as the base (a base-five number system has 5 symbols: 0, 1, 2, 3, 4).
3. Each has a set of fixed values for the places. Figure 9.6 shows the place values in base five.
4. Each has zero.
5. Each has a decimal point.

In addition to our base-ten system, two other bases are frequently used—base two and base sixteen. Both are currently important because of the way computers (and calculators, too) represent numbers internally.

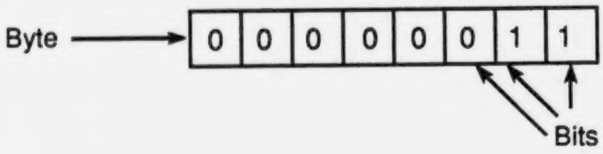
Inside the computer, the smallest unit of memory is called a *bit*. A bit is either on or off—if it is on it represents *one* and if it is off it represents *zero*. Eight bits to-

FIGURE 9.6
Base 5.



gether form a byte in computers like the Apple II. A byte is used to represent a single character like the letter *A*. It may also represent a small number. Figure 9.7 shows how the number 3 would be internally represented by an Apple computer.

FIGURE 9.7
The internal computer representation for 3.



To understand what these patterns of bits mean, you need to know something about the base-two numeration system. This system has only two symbols, 0 and 1. Each place in a numeral is a power of 2 (rather than a power of 10 as in the base-ten system):

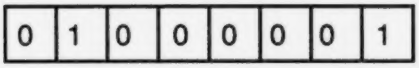
$$\begin{array}{cccccccccc} \ldots & 2^8 & 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ & 512 & 256 & 128 & 64 & 32 & 16 & 4 & 2 & 1 \end{array}$$

The base-two numeral 10111001_2 is equal to the base ten numeral, 185. Base two converts to base ten as follows:

$$\begin{aligned} 10111001_2 &= (1\times 2^7)+(0\times 2^6)+(1\times 2^5)+(1\times 2^4)+(1\times 2^3)+(0\times 2^2) \\ &\quad +(0\times 2^1)+(1\times 2^0) \\ &= 128 + 0 + 32 + 16 + 8 + 0 + 0 + 1 = 185 \end{aligned}$$

When a computer stores an alphabetic character internally, it uses a numerical representation for each letter, called the ASCII code (ASCII stands for "American Standard Code for Information Interchange"). The number that corresponds with the letter *A* is 65; therefore the internal representation for an *A* on computers like the Apple is as shown in Figure 9.8.

FIGURE 9.8
The internal computer representation of "A."



This corresponds with the number 01000001_2. The base-ten number that corresponds with 010000001_2 is 65:

$$\begin{aligned} 01000001_2 &= (0\times 2^7)+(1\times 2^6)+(0\times 2^5)+(0\times 2^4) \\ &\quad +(0\times 2^3)+(0\times 2^2)+(0\times 2^1)+(1\times 2^0) \\ &= 0 + 64 + 0 + 0 + 0 + 0 + 0 + 1 = 65 \end{aligned}$$

The ASCII code system has been agreed upon by computer manufacturers and scientists so that different kinds of computers can talk to each other (i.e., exchange data)—imagine the confusion if every brand of computer had its own internal language. Under the ASCII code, however, computers internally represent *B* by a base-two 66, *C* with a base-two 67, and so on.

Base Sixteen

Base sixteen is often used in place of base two when programmers need to converse directly with the computer in its native language (rather than an inefficient "high-level" or human-oriented language like BASIC or Logo, or through a software package) because the human mind has difficulty comprehending long strings of zeroes and ones. (See Figure 1.4R.)

The base-sixteen numeration system has 16 symbols and each place is 16 times its neighbor to the right. Table 9.2 shows the 16 symbols and their base-ten counterpart. The place values in a base-sixteen system are shown in Figure 9.9.

TABLE 9.2 Base sixteen symbols

Base 16	Base 10	Base 16	Base 10
0	0	8	8
1	1	9	9
2	2	A	10
3	3	B	11
4	4	C	12
5	5	D	13
6	6	E	14
7	7	F	15

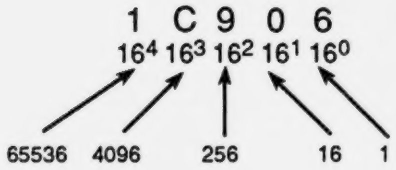


FIGURE 9.9
Base 16 place values.

The base-sixteen system is very useful in representing the internal workings of a computer because each group of 4 bits in a byte can be represented by a single base-sixteen digit. See Table 9.3.
Therefore, base sixteen for the computer code for 65 (remember, this is the code for the letter *A*) is 41_{16}.

$$\underline{\underbrace{0100}_{4}\underbrace{0001}_{1}}_{2}$$

To verify that this is correct, lets convert 41_{16} back to base ten. Remember, the place value of the second digit from the right in a base-sixteen number has a place value of 16^1.

$$41_{16} = (4 \times 16) + (1 \times 1) = 65$$

TABLE 9.3 Base 2 to Base 16 Conversions.

Base 16	Base 2
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

Let's try another one! We will convert $3FA_{16}$ to base ten.

$$\begin{aligned} 3FA_{16} &= (3 \times 16^2) + (F \times 16^1) + (A \times 16^0) \\ &= (3 \times 256) + (15 \times 16) + (10 \times 1) \\ &= 768 + 240 + 10 = 1018 \end{aligned}$$

A question still being debated is whether we should attempt to teach base two, five, or sixteen to children. During the "new math" era, many math textbooks included such instruction. The theory was that it would help children's understanding of our own base-ten system. Many children, however, had difficulty with these topics and research did not verify that such coverage actually improved children's understanding of numeration and related topics.

With the explosion of computers in the classroom comes a renewed interest in other bases, especially base two and base sixteen. One can argue that the vast majority of people will never need to know how the computer works internally, nor will they ever need to be able to program computers in machine language. Nevertheless, this topic appears in many elementary math textbooks and school districts' lists of objectives and is appropriate for selected students. Also, the author feels that *teachers* should understand other bases so that their understanding of numeration will be broadened.

ACTIVITIES FOR TEACHING NUMERATION

The following are a series of activities for developing the ideas associated with numeration. When children are being introduced to numeration for the first time, the teacher should first use many of the physical manipulations with two-digit numerals, then move to symbolic activities with these numbers. Next, move from physical models to symbolic activities for three-digit numbers. For those children who have already had some exposure to numeration, the teacher should select from among activities based on their level of achievement.

Bundled Stick Activities (Objectives A–C)

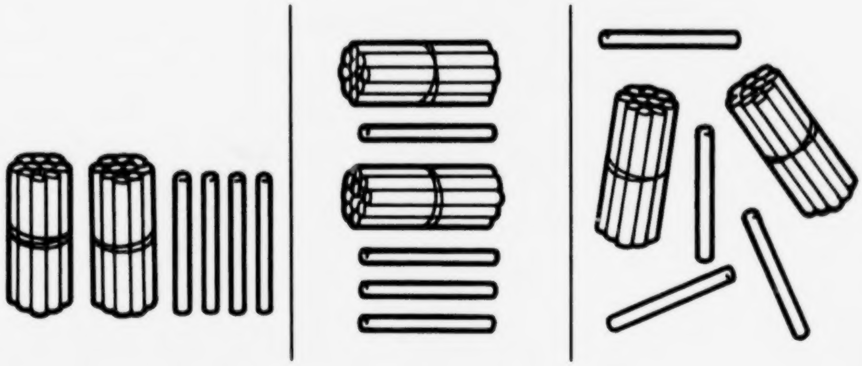
The previous chapter suggests using counting sticks for teaching the concept of *number* and single-digit number ideas. These same activities can be expanded to include beginning multidigit number ideas. It is suggested that bundled sticks be used to introduce two- and three-digit numbers, but that for numbers beyond 399, other models be used. Bundled sticks become cumbersome beyond this point.

Bundled sticks are used to represent multidigit numbers by grouping them into bundles of tens and hundreds. Popsicle sticks—where tens and hundreds are made by wrapping them with rubber bands—are perhaps the most convenient material to use. Tongue depressors, plastic drinking straws, or wooden dowels also work. Strips of tagboard or construction paper may also be bundled; however, these are not as durable. The important feature of these materials is that they be easily manipulated by children.

Bundled sticks offer a way to bridge the gap between using single objects to literally represent numbers like 53 and the abstract representation of a place-value system. Activities with the bundles help the learner realize that a single name (and later a single digit) can represent a group of objects. For example, the teacher might represent the number 25 with the sticks and ask the child, "How many tens are there?" This is an important step since the child has just learned that *3* means three objects. Suddenly, however, *3* represents three larger groups. If not developed properly, children may be confused by the equivalence of *1 ten* and *10 ones*.

It is important to realize that no place value is involved in the bundled-stick model; the positioning of sticks makes no difference in the value of the number being represented. The 3 positionings of sticks in Figure 9.10, for example, all represent the same number. The goal of instruction with the bundled sticks is to prepare the student for more abstract models where position *does* determine the value as with multidigit numerals.

FIGURE 9.10
Bundled sticks.



■ Put a handful of sticks on the child's desk. Make a bundle of 10 and wrap it in a rubber band. Ask the student to continue this process until no more bundles of 10 can be made. Ask questions about the final product. "How many (bundles of) tens are there?" "How many ones are left over?" "What is the name for this number (pointing to the sticks)?"

- Similar activities can be done with groups of children. Give each child a handful of sticks. Have them bundle as many bundles of 10 as possible. Ask them to think, but not say, how many tens, and ones. Then ask these questions to individual children.

- Represent a two-digit number with the sticks. Ask the child to say the appropriate number. Ask how many tens and ones there are.

- Say a two-digit number. Ask the child to represent it with the sticks. Do this activity first without pre-bundled sticks, requiring that children count out and bundle the sticks. Later, make bundles of 10 available so that children do not have to count them out. If a child must count the number of sticks in a bundle of 10 each time he represents a number, he needs more practice with earlier activities.

- Ask children to make models for numbers with a given number of tens and ones. For example, "Show me the number with 7 tens and 5 ones." Vary the order in which the tens and ones are given.

- Have children make and compare models for numbers whose names sound alike. For example, "Make the number 14 for me." "Now make the number 40." "How many tens does 40 have?" Etc. Practice with numbers in which the digits are reversed (e.g., 16 and 61).

- Tell a story about grouping by tens. For example, you might tell a story about a broomstick maker whose business was so good that he started selling his sticks in bundles of 10. He then kept track of the bundles of 10 he sold (*5* represents five bundles of 10). This made life much simpler for him. While telling the story, manipulate the sticks appropriately. Of course, a good storyteller will add many embellishments. Later, when introducing hundreds, the story can be expanded to ten bundles of 10.

- Place 250 or so individual sticks on the table and have the child make bundles of the tens. If you are working with groups of children, have them work together. If you are working with the child individually, you may want to help her. Introduce grouping by hundreds (10 bundles of 10). Group the sticks as simply as possible until no further bundling can be done. Ask questions about the resulting arrangement of sticks as was done with the two-digit numbers.

- Ask the child to show numbers with a given number of hundreds, tens, and ones. Have prebundled sticks available. For example, "Show me three-hundred five."

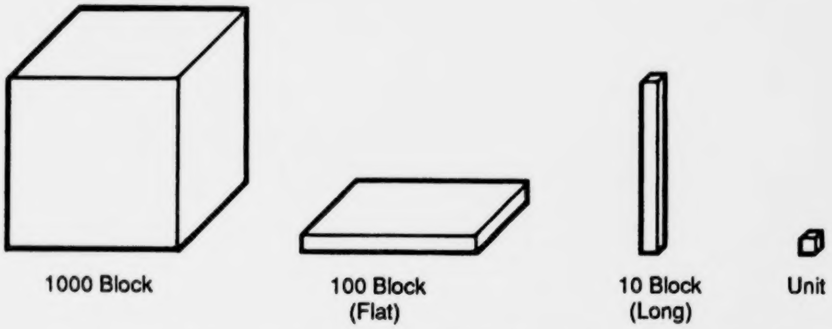
- Represent various numbers with the sticks. Ask the child how many hundreds, tens, and ones there are. Have the child state the number represented by the sticks.

Base-Ten Blocks (Objectives A–D)

Base-ten blocks (often called *numeration blocks*) are excellent materials for teaching many mathematical concepts, including numeration, operations on whole numbers, operations on decimals, and the metric system. Although they are somewhat expensive to purchase, they are by far the best overall manipulative you can buy for teaching elementary mathematics. They consist of a unit cube one cm on a side, a "long" whose dimensions are 10 cm by 1 cm by 1 cm, a "flat" which is 10 cm by 10 cm by 1 cm, and a "large cube" which is ten cm on a side. See Figure 9.11.

The major difference between the base-ten blocks and bundled sticks is that *ten, hundred,* and *thousand* consist of single blocks rather than groups of single objects. When doing regrouping activities, 10 units must be exchanged for a ten and 10 tens must be exchanged for a hundred. Of course, if a child doubts that the ten is equivalent to 10 units or the hundred equivalent to 100 units, they can be stacked next to each other for verification. With well-made base-ten blocks, 100 units or 10 tens will have exactly the same shape and volume as a hundred block.

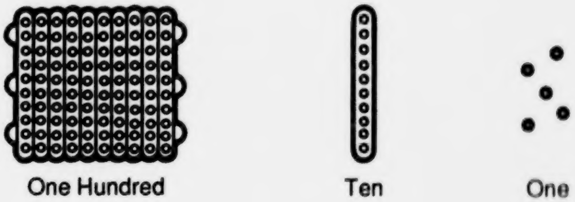
FIGURE 9.11
Base-ten blocks.



Activities with base-ten blocks should be similar to those described above for bundled sticks—children should represent verbally-presented numbers, should state the numbers represented by the teacher or other students, should state the number of tens, hundreds, etc. in numbers represented with the materials, and represent numbers with a given number of ones, tens, and hundreds. You may also wish to represent numbers with the blocks and have children write the corresponding numerals, or vice versa. On the other hand, you may wait to introduce written multidigit numerals until children have been exposed to the place-value models, such as the abacus.

If you find that base-ten blocks are too expensive, you may wish to consider a good, teacher-made (or pupil-made) alternative: bean sticks. Materials consist of dried beans (dried garbanzo beans work the best, but white beans or lima beans can also be used), and popsicle or stirring sticks. Glue ten beans on a stick to make a ten and 10 tens (with 3 sticks across the back) together to make a hundred. (See Figure 9.12.) Use them as you would numeration blocks. The major limitation of bean sticks is that it is difficult to make a model for a thousand. Also, as the author discovered when he was working on his dissertation, if there is a mouse in your classroom, your bean stick collection may be depleted.

FIGURE 9.12
Bean sticks.



Colored Chips (Objectives A–D)

Colored chips can be used to teach a variety of mathematical ideas, from numeration to division. In teaching place value, a color takes an agreed-upon value. For example, white could take a value of one, blue could have a value of ten, red of one hundred, and green of one thousand. The choice is arbitrary, just as the choice of a value for a position in a written numeral is arbitrary. In order to point out to the learner that the choice of colors for certain values is arbitrary, it might be a good idea to change one of the colors for a while during instruction (for example, use yellow for *ones* instead of white during a session).

The colored chip model is more abstract than the bundled stick or base-ten block models. With the bundled sticks, the student can unbundle the groups of ten or hundred to verify that the assumed number of sticks are there; with the base-ten blocks, the volume of a ten is 10 times that of a unit. With the colored chips, however, the child must remember an agreed upon color-value relationship.

A variety of different materials can be used for colored chips. Poker chips, circles cut from construction paper, and buttons are but a few of the possibilities. If you are working with an entire class, you might want to consider chips that are not noisy when manipulated, such as those made from construction paper or the ones sold by educational materials companies that are made from soft plastic.

When teaching place value, money should *not* be used as a model for multidigit numerals. There is no consistent relationship between the size of coins and bills and their value (indeed, a dime is *smaller* than a penny). Moreover, coins and bills are not powers of ten. Money should be used later as an *application* of whole number and decimal place value.

■ Determine the value of the various colors of chips, starting with only two colors. To point out that the choice of colors is arbitrary, you might want to have the learner decide which color will have a value of 10 and which will have a value of 1.

■ Take a handful of chips and place them in front of the child. Tell her that you want to represent the chips as simply as possible, and that every time there are ten white chips, you are going to exchange them for a blue chip. Help her through several examples, then ask her to do some examples without your help.

■ Represent a two-digit number with the chips. Ask the child to name the number. Ask how many tens and ones it has.

- Say a number and ask the child to make a model of the number using the colored chips.

- State a given number of tens and ones. Ask the learner to make a model that has that many tens and ones. For example, "John, please make a number (with the chips) that has 2 ones and 7 tens." Vary the order in which the tens and ones are given so that he will listen to and think about the place values you state rather than associating by rote blue with the first value and white with the second value. (This often happens in problems like "348 = ______ *hundreds* + ______ *tens* + 8 *ones*." If the order of the places is never varied, many children will copy by rote the first digit into the first blank, the second into the second blank, and so on. Varying the places requires them to think about the place values in the number.)

- Show the child how to regroup using the colored chips. Show him a model for a number like 35. Say, "What number is this?" When a child responds correctly, say, "Now we are going to show 35 another way." Exchange 1 ten for 10 ones. Ask the child how many tens and ones there are. Ask him to change the number back so that it is right.

- Represent a number like 81. Ask the child to change the chips so that you could take away five white chips.

- Place between 15 and 25 blue chips and about 30 white chips in front of the child. Ask her to make all the exchanges she can. If you have not already introduced the hundreds chip, try to lead the child to stating that ten blues should be exchanged for a red. For example, "Look at all these blue chips. Don't you think we could show them in a different way? What did we do when we had 10 ones?" Use different combinations of blues and whites and have the child simplify as much as possible.

- Represent numbers with hundreds, tens, and ones using the chips. Have the child state the verbal number associated with the chips. Have him name the number of tens, hundreds, and ones.

- Have the child represent a number stated by you. State a certain number of ones, hundreds, and tens, sometimes leaving out one or more of the places (e.g., "I would like for you to make the number for me that has no tens, no hundreds, and 1 ones."). Ask the child to make that number.

- Represent a three-digit number. Ask the child to change the number so that you could take away a certain number of tens or ones where that number is greater than the number of chips of that value. For example, represent 345 with the chips, then ask the learner to change the number so that you could take away 9 tens (or nine blues).

- Carry out similar activities for four-digit numbers, not exceeding 2500.

Other Bases

Activities similar to those described for bundled sticks and colored chips may be done with bases other than ten. If you are using colored chips, for example, to teach base five, the white chips could represent *units*, the blues the *fives*, the reds the *twenty-fives*, etc. Multibase blocks are similar to base-ten blocks but allow you to represent numbers in base two through base ten. For example, in base two, the "long" is two centimeters long, the "flat" is two by two by one centimeter, and the "large cube" is two centimeters on a side.

Place-Value Chart Activities (Objectives E–F)

At last we have a model that closely resembles our written numerals—the place-value chart. In the place-value chart, chips, buttons, beans or other small objects are placed on regions of a chart representing different values. However, the type of place-value chart we are referring to does not have its places labeled. If the places were labeled, any place-value arrangement would do, and we would still be able to determine the value of a number simply by reading the value of the place. Instead, we want the child to *remember* the place-value relationship as we do with written numerals.

Since written numerals have a right-to-left order of increasing value, the best procedure is to use that same order with a place-value chart. Your place-value charts can be 5″ by 10″ sheets of paper divided up into three or four regions of equal area, as shown in Figure 9.13. Other possibilities are to place slips of paper in "pockets" attached to poster board, or place metal washers or plastic chips with holes on pegs of an open abacus. (See Figure 9.14.)

FIGURE 9.13
A place-value chart.

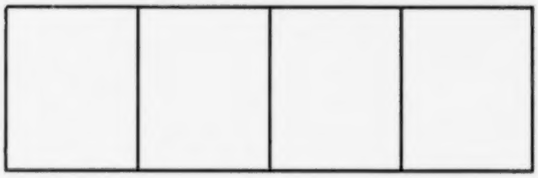
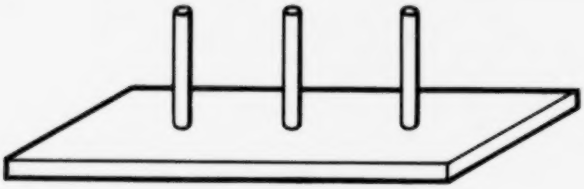


FIGURE 9.14
An open abacus.



All these devices use the same model for numerals: the place-value model. The important feature is that the objects used to represent the digits should be indistinguishable from each other; that is, they should all be the same shape, size, and color. The only way the child should be able to tell the difference between hundreds, tens, and ones is by their relative position on or in the place-value device.

Beginning activities with the place value chart are similar to those with the previous materials—children name the numbers represented by the materials and represent numbers with the materials. Later activities include naming the place values of numbers represented with the materials, writing numerals that correspond with the numbers represented by the materials, and making models for written multidigit numerals.

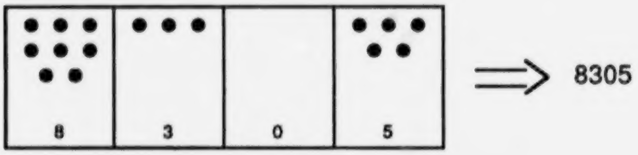
- Represent various numbers on the chart. For the first few, discuss the number of tens and ones, the relative locations, the materials used, etc., with the students. Have students determine and state how many ones, tens, hundreds, and thousands there are. Ask them to say the number represented on the chart.

- Say numbers up to 2500 and have students represent them on their place-value chart.

- Start with two-digit numbers and have students write the appropriate numeral below each place on the place-value chart. Be sure to include zeros. Have students write the numeral in condensed form on another piece of paper. This is an important step.

- After students have mastered two-digit written numerals based on the physical model, introduce three-digit numerals and later four-digit numerals. (See Figure 9.15.)

FIGURE 9.15
Place-value chart representation of 8305.



- Show students a numeral on a flashcard, or write it on the chalkboard or on a sheet of paper. Ask them to make a model of it with the chips and place-value chart. Discuss the results.

- Begin the process of separating the physical model from the written representation. For example, demonstrate a number with the place-value chart, then write the numeral on a separate sheet of paper. Remember, the goal of these activities is to lead children to an abstract understanding and ability to use numerals.

- At this point it might be good to go back and relate numerals to the bundled sticks and colored chips. Write numerals and have children represent that numeral with the bundled sticks or chips. Represent a numeral and have students write the corresponding numeral.

Some final notes about physical models for multidigit numerals: most children will not need to experience all of the models described above. At a minimum, children should work with either bundled sticks or base-ten blocks (probably both) and the place-value chart. Some children may need to work with all the models to gain understanding of multidigit numbers. Finally, do not move too quickly from manipulations with models to symbolic-only activities for numeration. The solid foundation of numeration understanding built with manipulatives will be the child's asset throughout the elementary grades.

Pictorial Activities (Objectives H–J)

Pictures are an intermediate step between physical models for numerals and totally symbolic representations. The purpose of using pictures is to help students form and manipulate mental images and progress toward a more mature level of learning where physical objects are not necessary for understanding—where ideas can be introduced through illustrations in books or worksheets, and where the child has enough experience to manipulate the pictures mentally.

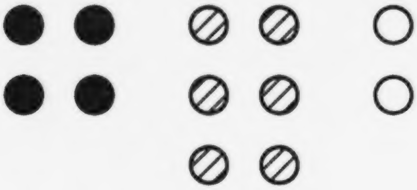
A major difference between pictorial and physical models for numbers is that physical models can be manipulated and pictorial models, for the most part, cannot. For example, you cannot bundle or unbundle a picture of sticks, nor can you exchange a picture of a red chip for a picture of ten blue chips. Arrows and such are sometimes used to indicate manipulations, but, in general, pictures are static representations of mathematical ideas. For this reason, with younger, less experienced children, it is necessary to have them first manipulate the physical models, then encounter the pictorial representations.

An advantage of having students experience pictorial activities is that they can be used in written tests. While it is easy to test an individual child or a very small group of children with manipulatives—having them make models of numbers or respond to questions about models made by the teacher—it is much more difficult to do so with a large group. Tests with pictures of manipulatives are easy to administer to large groups.

The amount of pictorial activity needed by students depends on their maturity. Certain children may need to be introduced to pictorial activities in conjunction with physical manipulations. With others it may be possible to simply introduce these activities separately.

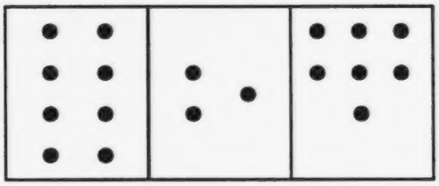
In general, instruction with pictorial models should progress from models that are like the physical models the students have worked with, to models that are more abstract. For example, pictures can be drawn of bundled sticks, colored chips, or chips on a place-value chart. (See Figure 9.16.)

FIGURE 9.16
A pictorial model for numerals.



■ Activities with these models can be similar to those with the physical models. For example, draw models for several three-digit numbers using dots on a place-value chart, as shown in Figure 9.17. Ask students to determine the number of ones, tens, and hundreds in the number and circle the correct answer.

FIGURE 9.17
A pictorial place-value chart.



Circle the number of tens, ones, and hundreds.

Tens 3 30 300

Ones 7 700 70

Hundreds 8 80 800 837

More abstract pictorial representations of numbers might involve agreed-upon symbols for given values. For example, | could represent ones, ▯ tens, and ☒ hundreds. As with physical models, the ultimate goals of instruction should be to develop understanding of place value. In the later stages of instruction, pictures of place-value models can be introduced; then activities can be carried out with the pictures. For example, you might draw dots on a chart to represent two- or three-digit numbers, place felt circles on a feltboard to show numbers, or draw lines on an abstract picture of an abacus.

Symbolic Activities (Objectives K–T)

Symbolic activities with multidigit numbers should be designed to reinforce and expand concepts already learned at the concrete and pictorial levels. The ultimate goal is to abandon the models completely and work entirely with numerals. Students should be taught to read and write large numbers; state the number of *ones, tens, hundreds, thousands,* etc. in a multidigit number; and state the value of each place in the number. They should be able to determine its total value, write it in expanded and exponential notation, and be able to regroup from one place to another. A student who possesses these skills will have an understanding of multidigit numbers and the foundation for later work with operations on numbers. And once the student has learned these concepts for three- and four-digit numbers he should be able to extend much larger numbers.

- Say numbers and have the student write them. With students who are just beginning to move beyond manipulatives, begin with numbers less than 350. With more advanced students, you may go well beyond the millions.

- Ask students to write numerals with a given number of ones, tens, hundreds, and so on. Later, mix the places so that they are not always in ascending or descending order. Show them written numerals and have them state the value of each place in the number.

- Teach the students to write numerals in expanded notation. Begin by writing the places as words. For example, 167 means 1 hundred, 6 tens, and 7 ones. Next, this can be expanded to a number/word sentence:

2371 = 2 thousands + 3 hundreds + 7 tens + 1 one

Problems for students could involve translating numerals to sentences such as those above or filling in missing digits or place values. Places can and should be mixed up after the beginning stages. For example,

30,057 = ____ ones + ____ hundreds + 0 thousands + 5 ____ + 3 ____

- The next step is to translate the verbal names for the places to numerals and demonstrate that multiplication is implied. This is a very large step since students may have had limited exposure to multidigit multiplication or may not have completely understood the concepts and skills involved. At this point you might be wise to go back to a physical model like the popsicle sticks or bean sticks. For example, 169 with sticks would be 1 bundle of a hundred, 6 bundles of ten, and 9 single sticks. Ask how many single sticks are in a bundle of one hundred. As you discuss the problem write it out as follows:

1 × 100 beans + 6 × 10 beans + 9 × 1 bean = 169 beans

Do various problems like this, perhaps using other models.

- Although money is not a good model for initial learning about numbers, it can be used after children have had ample experience with appropriate models to spark interest in and provide further examples of numeration concepts. For example, you might look at a certain number of dollars, dimes, and pennies (each less than 10) and write the amount of money in expanded notation, using written names for the coins or bills. Three dollars and sixty-one cents can be written as

3 dollars, 6 dimes, 1 penny

See if the child can come up with a way to find out how many pennies are represented by that sum. Later, write out the problem in expanded notation and discuss.

$$\overset{\text{pennies}}{3 \times 100} + \overset{\text{pennies}}{6 \times 10} + \overset{\text{penny}}{1 \times 1} = \text{the number of pennies}$$

- Give students 2 through 5 digit numbers (depending on their experience with numerals) and ask them to represent the numbers in expanded notation. Next, have them use Logo or another language to determine if their representation is correct by entering the expanded notation version of the number after the PRINT statement. Note that the parentheses are optional.

```
PRINT (5*1000)+(3*10)+(7*1)
```

- Have students do a variety of problems converting written numerals to expanded notation and vice versa. Problems can be straight conversions or they can involve mixed places as in the example above with 30,057.

- One way for students to use their calculators is for recording numbers presented in expanded form. For example, ask students to enter a number that has 3 tens, 5 hundreds, and 0 ones. The problems can be given in either written or verbal form. Have each student show you or a classmate the result.

- Another possibility is to have the student check to see if the expanded equivalent for a number is correct. For example, if she thinks that $368 = 3 \times 1000 + 6 \times 100 + 8 \times 10$, have her enter that expression. Some calculators allow the user to enter expressions like those above from left to right, then to push the = button and obtain the answer. Others require that the problem be done in pieces by calculating the products and summing them in the memory.

- In preparation for decimal numeration, the teacher can introduce exponential notation. It will first be necessary to introduce the notion of powers. For example, 2^4 means $2 \times 2 \times 2 \times 2$. Instructional development and student activities can be similar to those with expanded notation.

- In our modern society, we are continually bombarded by talk of billions of dollars spent by the government, tens of thousands of people in a football stadium, or millions of hamburgers sold, yet few people understand large numbers. Experiments done in math class can help give children a better understanding of the meaning of these numbers. Activities might involve determining the number of objects in a container, like the number of pieces of popcorn in a quart jar. Another example might involve determining how many pages from a telephone book it would take to make a stack 100 feet high. Experiences like these can also involve estimation where the student estimates a given, large quantity, then determines how close the estimate is to the actual number of objects. Obviously, a calculator would be useful here.

- Ask the student to enter a multidigit number in his calculator. Next, ask him to "remove" a given digit from the display; in other words, ask him to make that digit zero. For example,

> Enter the number 36,298. Now, without changing any of the other digits, remove the 90 (or 9 tens).

■ A good way to help point out the place/value relationships of numbers is to use the calculator alphabet (see Chapter 3). Write the calculator alphabet on the board. Ask the student what number spells "go" when turned upside down. "What number spells 'bell'?" Ask them to find other numbers that spell words. Ask them what word is made from a number with 4 ones and 6 hundreds.

■ The + key can be used to investigate number patterns. For example, young children can push 1 + 1 + etc., saying each time what number they think will appear on the display. Older students can do the same with tens, hundreds, or non-multiples of 10, perhaps using the – key as well.

Regrouping Activities (Objective U)

Regrouping activities provide important preparation for operations on whole numbers. Children should be taught first to regroup with manipulatives, then to do corresponding activities with symbols.

■ Have the student take a handful of base-ten blocks (or colored chips) that includes units, tens, and hundreds. Ask how many of each there are. Next, have her make exchanges so that she has no more than 10 units, tens, or hundreds. Again, ask how many of each shape she is holding. Discuss the fact that the same number or amount is represented before and after the exchanges. If the student needs further verification that the same quantity is being represented before and after the exchanges, repeat the activity with two sets of numeration blocks. Start with 14 units, 16 "longs," and 2 "flats" in each set. Stack the two sets to make the same shape. Now, do the exchanges for one of the sets and restack that set to verify that the two have the same volume. (See Figure 9.18.) Repeat with various-sized groups of materials.

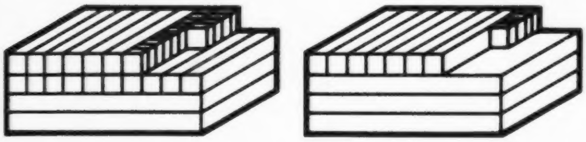


FIGURE 9.18
Verifying that regrouped sets are equivalent.

■ Give students sets of base-ten blocks and ask them to represent 135 or other, similar numbers. Discuss the number of ones, tens, and hundreds. Have them regroup so that the same total amount is represented but in a different way (e.g., some students will exchange a ten for 10 ones, others will exchange the hundred for 10 tens). Discuss the results and have them make further exchanges, each time discussing the results and the fact that you have the same amount as you started with. Now, exchange in the other direction until there are no more than 10 of any shape. Ask the children if they have the same as they started with.

■ Have children regroup symbolically. For example, have the students do problems similar to the following using the chalkboard and/or on worksheets.

1. 305 = 3 hundreds, _____ tens, 5 ones
   or _____ hundreds, 10 tens, 5 ones
   or _____ hundreds, 20 tens, 5 ones
2. 1 thousand, 15 hundreds, 6 tens, 0 ones = _________ (1 number)
3. 21 hundreds, 3 tens, 18 ones = _________ (1 number)

Rounding Activities (Objective V)

Rounding activities are an excellent way to reinforce children's understanding of numeration. They focus children's attention on place value and related concepts. Furthermore, in today's technological world, the ability to round numbers to a given place is a basic skill. This skill is used in many of the mental computation and estimation processes discussed in Chapter 6 and in later chapters.

Perhaps the best model for developing rounding skills is the number line. Before children are asked to use the number line to round, however, you should do some groundwork.

- Make number lines where the tens are identified. Say or write a variety of numbers (e.g., 45, 123, etc.) and have the children locate the position of the numbers, as in Figure 9.19. Make a worksheet with number lines in tens, hundreds, and/or thousands without units and have children locate numbers. (See Figure 9.20.)

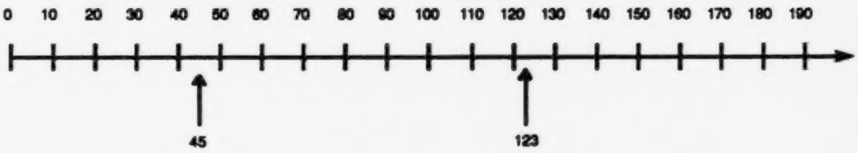
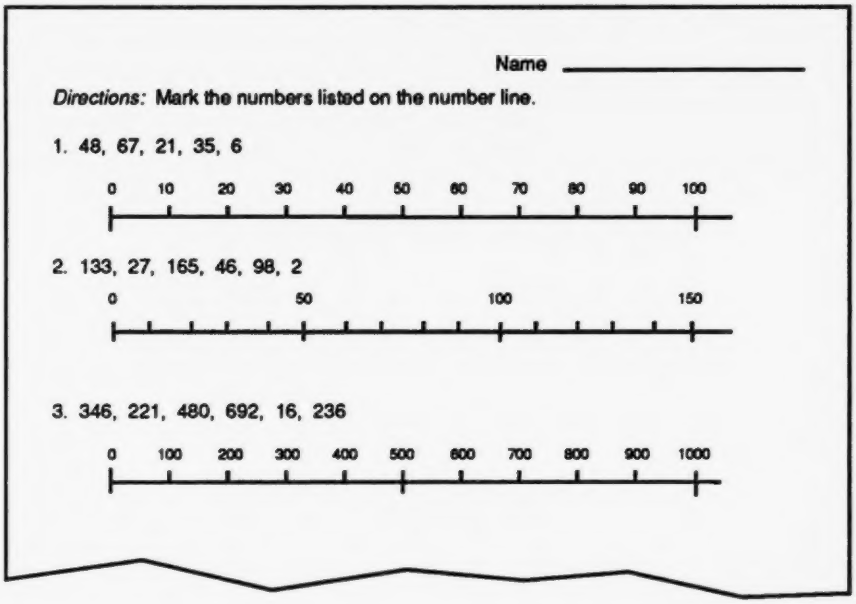


FIGURE 9.19
A number line in tens.



Name ______________________

Directions: Mark the numbers listed on the number line.

1. 48, 67, 21, 35, 6

2. 133, 27, 165, 46, 98, 2

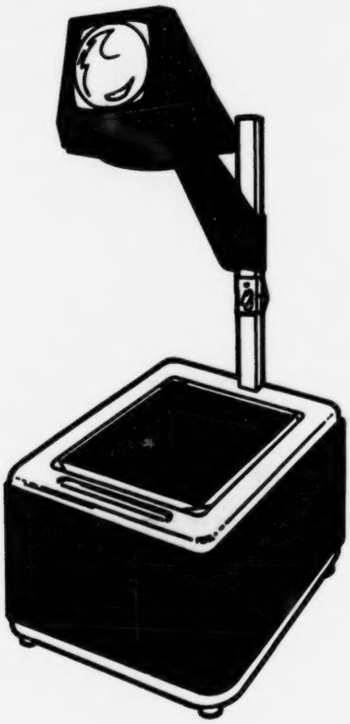
3. 346, 221, 480, 692, 16, 236

0
10
20
30
40
50
60
70
80
90
100
0
50
100
150
0
100
200
300
400
500
600
700
800
900
1000

FIGURE 9.20
A number line worksheet.

- Introduce rounding to the nearest 10 using a number line with tens marked and highlighted (with or without the units marked). Note: this would be a good lesson for the overhead projector (see Figure 9.21). Locate numbers like 43, 67, and 48 and discuss which ten they are nearest. You may wish to use a centimeter ruler to measure the distance between the number to be rounded and the 2 nearest tens.

FIGURE 9.21
Rounding using number lines on the overhead.



Discuss the convention for rounding numbers that are equidistant between 2 tens (e.g., 45, 65, etc.). Do similar activities involving rounding to the nearest hundred, thousand, and so on.

- Present the algorithm for rounding to a given place; if the place immediately to the right of the place you are rounding to is 5 or larger, round up (increase the place you are rounding to by one and change all places to the right to zeroes), otherwise round down (change all places to the right to zero). Of course, when presenting this algorithm, you should make it meaningful by relating it to your previous work with number lines.

- The calculator can be used to check to see if a number has been rounded correctly. For example, the student could use the calculator to subtract 949 from 1000 and 900 from 949 to determine if 949 rounded to the nearest hundred is 900 or 1000.

The length of this chapter is indicative of the importance of the topic of numeration and place value. Make sure that you give it the appropriate attention in your math instruction—it should be reintroduced at the appropriate level and with the appropriate materials every year, starting in the first grade. It is not unreasonable to spend three weeks on activities from this chapter in the first grade, two weeks in the second, and one or more weeks in subsequent grades.

SAMPLE DRILL AND PRACTICE ACTIVITIES

The following games are designed to provide the student with enjoyable drill and practice at the concrete as well as symbolic levels. It is hoped that they will provide the teacher with ideas for developing his or her own games designed to meet specific behavioral objectives.

Activity 9.1: Chingo

Objective: To provide practice in associating chip representations on a place-value chart to written numerals.

Materials:

1. Box
2. Numbers (10–399) on 1 cm. pieces of tagboard
3. Place-value chart and chips
4. Chingo cards (see Figure 9.22). Each card should contain a different arrangement of some of the numbers.

FIGURE 9.22
A CHINGO card.

CHINGO

40	16	257	372	19
80	60	15	18	110
100	17	FREE SQUARE	20	200
217	70	14	120	251
30	90	147	203	452

Methods:

1. Draw a number out of the box. Do not allow the pupils to see it.
2. Represent the number with chips on a (non-labeled) place-value chart. With large groups, use a transparent place-value chart on an overhead projector.
3. If the numerical representation of the colored-chip arrangement appears on a child's card, she places a marker over it.
4. Allow five or six bingos before clearing the cards and starting over.

Activity 9.2: Chip Trading Game

Objective: To provide concrete experience with regrouping.

Materials:

1. Place-value chart
2. White, blue, red chips
3. Dice containing numerals 0, 3, 5, 7, 8, 9.

Methods:

Variation 1

1. Give each child a place-value chart and a die. Have everyone start out with five blue chips (tens). Explain that when you have ten whites (ones), you exchange them for a blue (ten) and when you have ten blues you exchange them for a red (hundred).
2. Have each pupil roll his die, take the corresponding number of whites, and make any exchanges he can.
3. The first child to get a red chip wins. Continue the game until five children have exchanged for a red chip. Allow the winners to continue after they have a red chip (have them see how large a number they can get).
4. Ask questions about the numbers represented by the chips: the number of tens, ones, and hundreds; the relationship between the numbers represented on the place-value charts, etc.

Variation 2

Start out with a red chip and subtract the number of units corresponding to the value on the die. The first pupil to reach five blue chips is the winner.

Activity 9.3

Place-Value Cross Number Puzzle

Objective: Review of whole-number numeration concepts (see Figure 9.23).

SUGGESTED READINGS

Anderson, A. L. Why the continuing resistance to the use of counting sticks? *Arithmetic Teacher* 25(6), March 1978: 18.

Coplestone-Loomis, L. The big pumpkin count. *Arithmetic Teacher* 29(2), October 1981: 36.

Engelhardt, J. M., Ashlock, J. M., and Wiebe, J. H. *Teaching Children to Understand and Use Numerals.* Boston: Allyn and Bacon, 1984.

Skokoohi, G. Manipulative devices for teaching place value. *Arithmetic Teacher* 25(6), March 1978: 48–51.

FIGURE 9.23
A number puzzle.

1		2	■	3	4
5			6	■	
	■	7		8	
■	9	■	■		■
■	10			■	11
12			■	13	

Objective: To provide symbolic practice with numeration concepts.

Materials: Pencil and paper

Across

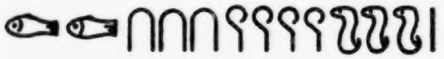
1. 800 + 20 + 5 = ____
3. A number that has 3 tens, 7 ones ____
5. (1 x 10) + (3 x 100) + (7 x 1000) + 3 = ____
7. A number that has 5 thousands and 3 tens ___
10. The place value of the 9 in 987 is ___
12. The total value of the 8 in 179, 846 is ____
13. 130, 120, 110, 100, ____

Down

1. (8 x 100) + (7 x 10) + 0 = ____
2. 10 + 5 + 500 = ____
4. 674 rounded to the nearest hundred ____
6. 6237 = 6000 + 200 + ____ + 7
8. How many tens are in 350, including the hundred's place?'
9. (1 x 10) + (8 x 100) = ____
11. The place value of the 6 in 3268 is ____

QUESTIONS

1. Do the conversions stated below:
   a. Write 23,500 in Egyptian numerals.
   b. Convert the following Egyptian numeral to base ten.



   c. Write 73,456 in Roman numerals.
   d. Write CCCXLIV in base ten.
   e. Convert 10110001_2 to base 10.
   f. Convert $8C_{16}$ to base 10.

2. Find the newest elementary mathematics textbook series you can and investigate how multidigit numeration is taught in the third-grade text. Next, do the same for a math series that is about ten years older. How do the two presentations compare?
3. Design a game puzzle for practicing multidigit numeration (see Chapter 20).
4. Discuss how the following error pattern exhibited by a fourth-grade student relates to numeration and how you could use manipulative materials to teach the numeration concept(s) that the student does not understand:

$$\begin{array}{r} 79 \\ +\ 14 \\ \hline 813 \end{array} \qquad \begin{array}{r} 46 \\ +\ 56 \\ \hline 912 \end{array} \qquad \begin{array}{r} 28 \\ +\ 17 \\ \hline 315 \end{array}$$

5. Write a computer program that accepts two inputs, the first a multidigit numeral and the second, one of the digits in the first numeral (e.g., 347 and 3). The program then prints out the place value of that digit in the first number (e.g., "HUNDRED").
6. Design a manipulative device different from the ones discussed in this chapter for teaching *place value* and related concepts.

CHAPTER

10

Addition & Subtraction of Whole Numbers

ADDITION OF WHOLE NUMBERS

The idea of addition is a simple one—joining sets together and determining the number of objects in the total set.

> *There are thirteen frogs sitting croaking on the banks of the pond. Eight frogs from another pond hop over and join in the chorus. Now how many frogs are croaking on the banks of the pond?*

Of course, addition becomes much more complicated when one is dealing with larger numbers or with fractions. If, for example, you want to add ½ of a pie to ⅔ of a pie, you must find a way to express your answer as a single fraction.

The standard algorithm that we spend so much time teaching our children is but one way to arrive at answers to addition problems. Suppose a political convention were held in a small town of 3457 residents. If none of the townspeople were out of town and if 1549 out-of-towners came for the convention, how many people would be in the town during the convention? There are many ways of solving this problem. One way would be to start at the number 3457 and count 1549 more, then note the number where your counting ended. Another method would be to enter the following into your calculator:

$$3000 + 1000 + 400 + 500 + 50 + 40 + 7 + 9.$$

Or, if we needed only an approximate answer, we might focus on the front digits in the two addends and estimate that there were approximately 5000 people there.

We need to use a large variety of materials and methods when teaching addition and other operations. We also need to remember that most people will use calculators to add long sequences of numbers or numbers with many places. Therefore, our instruction should focus on developing understandings, pencil and paper proficiency with smaller numbers, the ability to estimate sums, and the ability to solve problems with calculators and computers. The first section of this chapter looks at a variety of ways to develop understanding of the addition process and proficiency in finding answers to addition problems.

Behavioral Objectives for Addition of Whole Numbers

A. Join two given sets of objects (< 10) and count to determine the sum of the sets.
B. Represent each addend of a written basic fact (each addend < 10) with objects, join the sets and count to determine the sum.
C. Be able to state sums, within 3 seconds (addends from 0 to 10), when shown a flashcard.
D. Represent two addends verbally given (< 200) with bundled popsicle sticks or base-ten blocks, join the sets, regroup if necessary, and state the sum.
E. Represent two addends verbally given (< 5000) on a place-value chart or abacus and manipulate the materials to find the sum.
F. Solve verbally stated problems involving addition (addends < 100).
G. Add single-digit numbers to two-digit numbers with and without regrouping (vertical or horizontal form).
H. Add a number to another number with and without regrouping, using standard algorithm.
I. Add two numbers using left to right algorithm.
J. Add more than two numbers.
K. Give a reasonable estimate of the sum of any two multidigit numbers without doing pencil and paper computations.
L. Add two or more numbers using a calculator.
M. Add two or more numbers using the PRINT statement in Logo or BASIC.
N. Solve verbal and applied problems involving addition.

TEACHING ADDITION

Activities with Single Objects (Objectives A–B)

Pupils need to see that addition involves joining sets of objects. Eventually we will have children do addition exercises with pencil and paper, calculator or computer, or in the context of a verbal problem. But in each of these kinds of calculations, the idea that addition involves the joining of sets should be kept in mind.

■ Demonstrate to the student how to make sets of objects, then how to join them and determine the total. Materials may be any objects that children are able to manipulate easily like inch cube blocks, poker chips, popsicle sticks, or plastic animals. To demonstrate that you are adding one *set* to a second *set*, make and join distinct groups of objects (see Chapter 8). For example, single objects in each set could be placed in loops of yarn, small boxes, or cups, then the contents of the two groups could be combined into one loop or container to show the process of addition. Objects in the combined group can then be counted to determine the total. When doing these activities, be sure to accompany the actions with the correct language to prepare the pupil for more abstract activities. The following dialogue is an example:

> *Teacher:* "Let's make a group of blocks over here to show four. Let's make another group here to show five. Now, we are going to add (join) the two groups together. Can you tell me how many there are altogether?"

After your demonstration, encourage the student to do the manipulations for him- or herself.

> *Teacher:* "I would like you to make a group to show six. Now, make a group to show two. Now add them and tell me the sum."

Later, give the student the problem in 'usual' verbal form but encourage the use of the materials in the format above.

> *Teacher:* "I would like you to use the blocks to find three plus four."

Learning Addition Facts (Objective C)

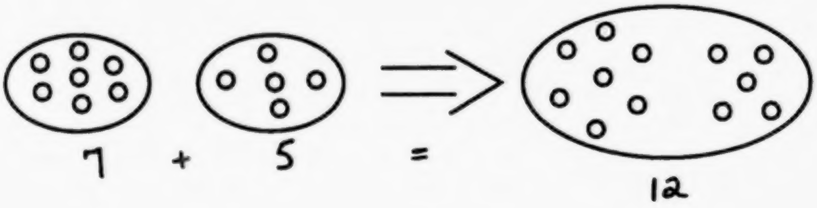
After sufficient verbal activities with materials, written numerals and number sentences can be associated with the manipulations. An appropriate intermediate step is to do pictorial activities where sets are represented and children write numerals to show the sums. Later, when students understand the concepts involved, drill and practice can help fix the basic sums in their minds.

- Represent sets to show addends. Have the pupils write a numeral for each set. Combine the sets. Have them write a numeral for the sum.

- For sums that are ten or larger, use popsicle sticks and make a bundle of ten sticks to show the tens and point out that the unbundled sticks represent the ones. Have the students write the addends and the sum as above.

- Give the students worksheets with pictures of objects showing sums (see Figure 10.1). Have them write the addends and the sum.

FIGURE 10.1
Pictorial addition.



- The old standby, flashcards, can always be used to help students memorize basic facts. Allow a given amount of time for them to give the right answer. As they become more proficient, decrease the time until the response is instantaneous. Certain events that occur in a period of a few seconds can be used to place a time limit on a student's response and to add motivation. For example, drop a rubber ball and show the flashcard simultaneously, requiring that the student give the correct answer before it strikes the ground a second time.

■ It is also a good idea for students to learn to respond quickly in writing to basic-fact problems. Children usually enjoy timed drills if they are rewarded and if the drills are geared to their ability level. To encourage memorization of the one-digit addition sums, you may wish to provide extrinsic rewards like stars on a graph for each group completed (e.g., the 6s), certificates, letters of congratulations to parents, free time during class, and so on.

■ One of the most motivational ways of providing drill is through one of the many computer math drill programs on the market. An example is "Alligator Mix," one of many that incorporate drill on the basic math facts into arcade-style graphics.

Appendix D of this book contains a math drill program that includes basic facts. It allows you to choose the range of the problems and the operation (e.g., adding single-digit numbers.

PROPERTIES OF ADDITION

Addition has three fundamental properties that are important when learning addition facts, when adding multidigit numbers, when doing mental addition, or when using addition in other operations such as multiplication. These unifying properties help us understand why certain procedures work, and at the same time, knowing them helps simplify the learning process because they are common to many mathematical procedures.

(1) The Identity Property for Addition

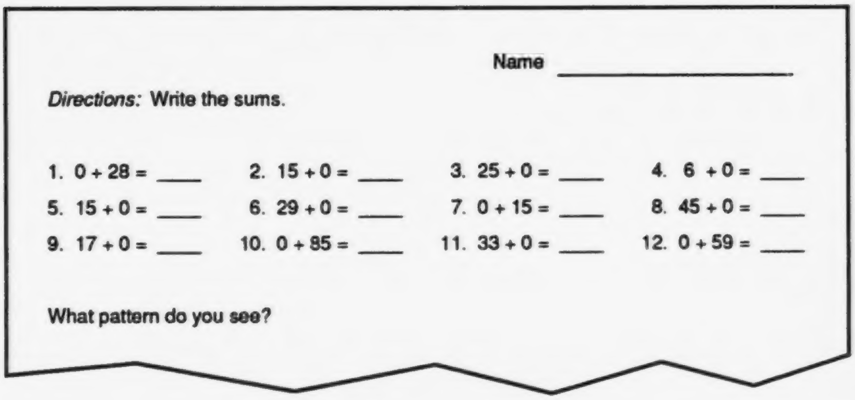
The *identity* for an operation is the value (number) that when combined with another number results in no change. In the addition operation, the identity is *zero* because when you add zero to any number the result is that number.

$5 + 0 = 5$
$0 + 45 = 45$
$2{,}335{,}507 + 0 = 2{,}335{,}507$

The identity property is important in addition because, once understood, children will not need to memorize facts involving addition of a number and zero. Even more important is the general idea of the identity that has applications in other operations, such as multiplication.

■ To develop the concept of the identity for addition, give children a worksheet containing a sequence of exercises involving addition of zero and another number (see Figure 10.2). You may wish to have the children use manipulatives, calculators, or computers to solve the exercises. If you decide to allow students to use the computer to solve the problems, you need to introduce the PRINT statement and its use in doing arithmetic computations. Of course, many students do not need to do the operations because they already understand the property. Others see a pattern after a few examples. The final step in this activity is to ask the pupils to state a generalization about adding zero to any number.

FIGURE 10.2
An identity worksheet.



Name ______________

Directions: Write the sums.

1. 0 + 28 = ___ 2. 15 + 0 = ___ 3. 25 + 0 = ___ 4. 6 + 0 = ___
5. 15 + 0 = ___ 6. 29 + 0 = ___ 7. 0 + 15 = ___ 8. 45 + 0 = ___
9. 17 + 0 = ___ 10. 0 + 85 = ___ 11. 33 + 0 = ___ 12. 0 + 59 = ___

What pattern do you see?

(2) The Commutative Property for Addition

Another name for the commutative property is the *order property.* It is called the order property because the order in which one adds two numbers does not affect the sum. If you have five marbles in a box and add three more, you have a total of eight marbles. Likewise, if you have three marbles in another box and add five more to them, you also have eight marbles.

$5 + 3 = 3 + 5 = 8$
$127 + 93 = 93 + 127 = 220$

The discovery of this property is a very important step in the process of computing answers to addition problems and in the mastering of basic addition facts. A strategy used by most young children in determining the sum of two values is *counting on*—the child starts counting at the first of the two numbers to be added, then counts on as many numbers as are in the second addend (keeping track of how far has been counted using fingers). For example, to calculate 5 + 3 the child will start at five and "count on" for three more numbers. Knowledge of the commutative property can be very useful when solving problems like 3 + 12 or 1 + 9—it is much easier to count three beyond 12 or one beyond 9 than to do the reverse. Also, in memorizing the single-digit addition facts, the knowledge that 8 + 7 and 7 + 8 are equivalent nearly halves the number of facts that need to be memorized.

(3) The Associative Property

The *associative* property of addition is based on the fact that addition is *binary.* A binary operation is one that is carried out on two operands. For example, when computing 3 + 4 + 5, we combine the addends two at a time: (3 + 4) + 5.

The associative property of addition states that when adding three numbers the sum is the same whether you add the first two then add the sum to the last, or add the last two before adding their sum to the first.

$(3 + 4) + 5 = 3 + (4 + 5) = 12$

This property is used explicitly or implicitly in our algorithms. For example,

$$\begin{aligned} 41 + 5 &= (40 + 1) + 5 \\ &= 40 + (1 + 5) \\ &= 40 + 6 \\ &= 46 \end{aligned}$$

When adding two multidigit numbers, we use both the commutative and the associative properties:

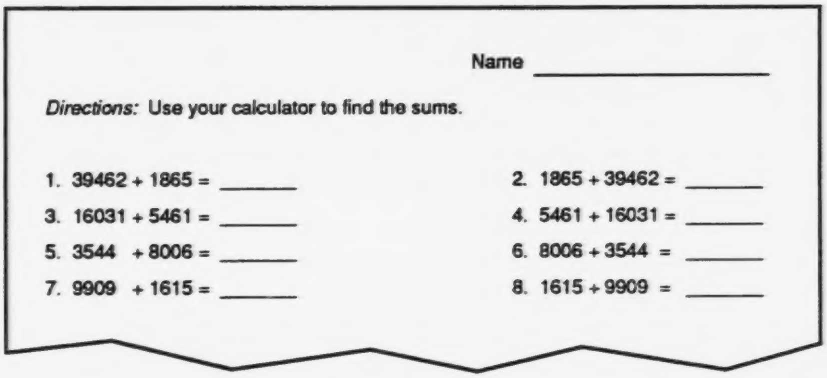
$$\begin{aligned} 45 + 23 &= (40 + 5) + (20 + 3) && \textit{Expanded Notation} \\ &= 40 + (5 + (20 + 3)) && \textit{Associative Property} \\ &= 40 + ((5 + 20) + 3) && \textit{Associative Property} \\ &= 40 + ((20 + 5) + 3) && \textit{Commutative Property} \\ &= (40 + 20) + (5 + 3) && (\textit{Skipping steps}) \\ &\;\;\ldots \end{aligned}$$

TEACHING THE ADDITION PROPERTIES

The process of teaching the identity and commutative properties is not difficult. Children may be led to discover both by doing properly arranged problems on worksheets and by then being asked to state a generalization about the problems.

■ Give students a worksheet similar to the one shown in Figure 10.3. Discuss the pattern in large- or small-group settings. Older children may be asked to write their generalizations. On the other hand, it may be more appropriate to tell students the associative property and have them do examples than to try to lead them to discover it.

FIGURE 10.3
Commutative property worksheet.



Name ____________________

Directions: Use your calculator to find the sums.

1. 39462 + 1865 = ______
2. 1865 + 39462 = ______
3. 16031 + 5461 = ______
4. 5461 + 16031 = ______
5. 3544 + 8006 = ______
6. 8006 + 3544 = ______
7. 9909 + 1615 = ______
8. 1615 + 9909 = ______

■ Place groups of five, six, and seven objects on an overhead projector or magnetic board. Write the problem

$$5 + 6 + 7$$

so that all may see. Place parentheses around the 5 and 6, then join the sets of five and six objects. Write the sum. Next, join the group of eleven objects to the group with seven objects and write the total.

$(5 + 6) + 7$
$11 + 7$
18

Repeat, this time joining the group of six and seven first.

$5 + (6 + 7)$
$5 + 13$
18

Discuss the generalization (the order in which you add two of three addends does not affect the answer).

- Have students complete pencil-and-paper problems involving the associative property. Typically, elementary textbooks include activities such as the following:

12 + 5 + 8 = (12 + ____) + 8
= ______ + 8
= ______

TEACHING MULTIDIGIT ADDITION

Bundled-Stick/Base-Ten Block Activities (Objective D)

Bundled-stick activities are an extension of the place-value activities from Chapter 9. They simply involve (1) representing each addend with the sticks, appropriately grouped, (2) combining the groups to show the operation of addition, (3) regrouping wherever necessary to express the sum in simplest terms, and (4) stating the sum. The problems and answers can be expressed either verbally or in writing. Figure 10.4 demonstrates adding a three-digit number, 145, to a two-digit number, 37, with regrouping.

PROBLEM: Add 145 and 37 (See Figure 10.4 on the next page.)

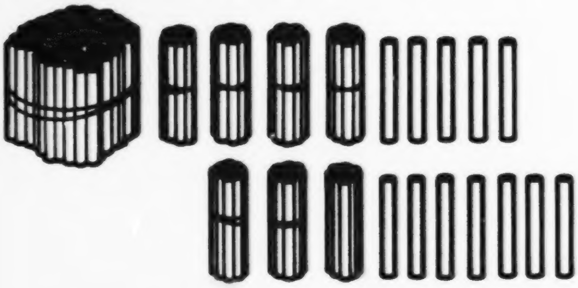
- After appropriate demonstrations, initial problems can involve presenting the sum of single-digit and two-digit numbers verbally and asking students to find the answer with the sticks and state the sum. Problems can be presented as follows: "I would like you to add nine and forty-seven using the sticks."

- Have the student add numbers up to 150 using the bundled sticks or base-ten blocks. State the sum verbally; have her do the appropriate manipulations and then state the answer verbally.

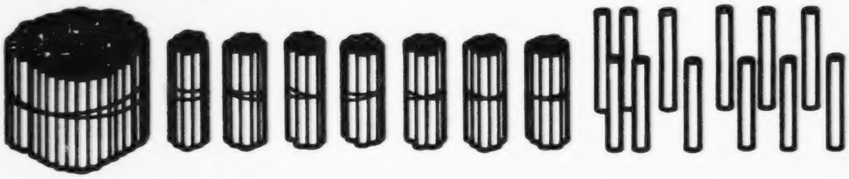
- Have the student solve written problems using the sticks or blocks. Have him do the manipulations to find the answer, then express it in writing in the appropriate place in the written problem.

FIGURE 10.4
Bundled stick addition.

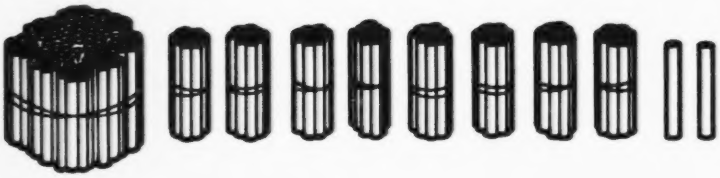
Step 1: Represents the addends:



Step 2: Join the steps:



Step 3: Regroup:



Step 4: Express answer: 182

Place-Value Chart Activities (Objective E)

Figure 10.5 demonstrates the sequence of manipulations for adding two multidigit numbers using a place-value chart. Preparation for this manipulation comes from the place-value activities in Chapter 9.

PROBLEM:

$$\begin{array}{r} 2056 \\ +\ 928 \\ \hline \end{array}$$

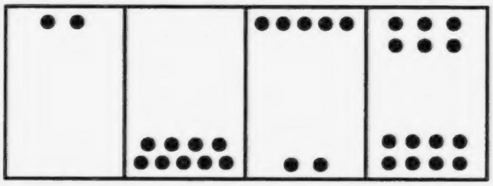
- Present the problems verbally and have the student use the materials to find the sum. The size of the addends should be determined by the level of problems being studied at the moment (for example, it would not be appropriate to solve four-digit addition problems before students have been exposed to four-digit numeration).

- Present problems in writing (both vertically and in number-sentence form). Have the student use place-value materials to solve the problem, then record the results in the appropriate place in the problem.

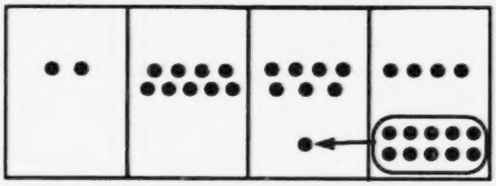
FIGURE 10.5
Place-value chart addition.

Problem: 2056
+ 928

Step 1: Represent addends:



Step 2: Combine and regroup:



Step 3: Express answer: 2984

EXAMPLE: Solve the following using the place-value chart

Record your answer here.

1. 245 + 78 = ____________________
2. 98 + 308 = ____________________

Etc.

■ Give a set of bundled sticks, numeration blocks, or colored chips (depending on the materials students have recently worked with) and a die to groups of 3 or 4 students. Each student starts with no sticks (or blocks or chips). Roll the die to see who goes first; each player then rolls the die in clockwise order and takes the corresponding number of sticks, adding it to whatever he or she has already accumulated. Whenever a player has 10 sticks, he or she makes a bundle of 10 with a rubber band. The first player to get a bundle of 100 (or a flat or red chip) is the winner. A variation is to use 2 dice—one color for ones and another for tens—and take materials that correspond with the amount tossed on the dice. In this variation, the goal should be 300 or larger.

Symbolic Activities (Objectives G–J)

In order to help students become proficient at adding multidigit numbers, it is important to develop their understanding of the overall process of multidigit addition and the substeps involved. The manipulations described here will go a long way toward achieving this end. Teachers can do much at the symbolic level to promote understanding. Indeed, one of the algorithms discussed below is almost as efficient as the traditional algorithm but is far more meaningful.

Using the Expanded Notation

The student's background in and understanding of numeration will help her grasp the following. It is also important that the student understand the associative property for addition. The purpose of this sequence is to show that in addition of multidigit numbers we add ones to ones, tens to tens, hundreds to hundreds, etc. The following sequence shows the symbolic sequence for adding two two-digit numbers and the accompanying manipulations with bundled sticks or base-ten blocks.

PROBLEM: $23 + 18 = ?$

	Symbolic Steps	*Manipulations*
STEP 1:	Write each addend in expanded notation:	
	$(20 + 3) + (10 + 8) =$	2 tens, 3 units plus 1 ten and 8 units
STEP 2:	Change the order of partial addends so that ones can be added to ones, etc.:	
	$20 + 10 + 3 + 8$	Rearrange the bundles and units
STEP 3:	Add tens to tens and ones to ones:	
	$30 + 11$	Combine units
STEP 4:	Write 11 in expanded notation:	
	$30 + 10 + 1$	Bundle ten units into one ten
STEP 5:	Add tens to tens:	
	$40 + 1$	Answer
STEP 6:	Simplify:	
	41	

It would probably be fruitless to ask most children to go through all the above steps on their own. (Note, however, that this sequence *is* feasible using the calculator—see below.) A better strategy would be to go through several problems and have the students follow along, then later have them state what the next step in a series would be or have them fill in blanks in the sequence.

EXAMPLE: "Fill in the blanks below."

$$\begin{aligned} 35 + 28 &= (30 + 5) + (20 + \underline{\qquad}) \\ &= (30 + \underline{\qquad}) + (5 + 8) \\ &= 50 + 13 \\ &= 50 + \underline{\qquad} + 3 \\ &= 60 + 3 = \underline{\qquad} \end{aligned}$$

Expanded notation in vertical form can also be used.

EXAMPLE:

$$\begin{array}{r} 139 \\ \underline{+\ 78} \\ \end{array} \qquad \begin{array}{l} 100 + 30 + 9 \\ \underline{\qquad\quad\ \ 70 + 8} \\ 100 + 100 + 17 = 200 + 17 = 217 \end{array}$$

Again, students can supply missing values after the teacher has gone through a step-by-step instructional sequence.

EXAMPLE:

$$\begin{array}{r} 139 \\ \underline{+209} \\ \\ \\ \end{array} \qquad \begin{array}{l} 100 + 30 + 9 \\ \underline{200 + \ \ 0 + 9} \\ \_\_ + 30 + \_\_ = 300 + 30 + 18 = 300 + 30 + \_\_ + 8 \\ \qquad\qquad\qquad\ \ 300 + 40 + \ \ 8 = 348 \end{array}$$

Listing Partial Sums

In the above sequences, the students are made aware of the overall process of addition, but it is easy to get bogged down in the mathematical manipulations. The following algorithm does not have that disadvantage. It is almost as efficient as the usual algorithm, yet the overall processes are clearly visible to students.

The *partial sums* algorithm simply displays each partial sum in full rather than combining adding and "carrying" into a single step. This avoids the complications many students experience when carrying, and also displays intermediate steps so that they are more meaningful. In this algorithm you add ones to ones, then write the sum; you add tens to tens, then write the sum in full; and so on. The following is an example:

PROBLEM:

$$\begin{array}{r} 1459 \\ \underline{+\ 526} \end{array}$$

SOLUTION:

$$\begin{array}{r} 15 \\ 70 \\ 900 \\ \underline{1000} \\ 1985 \end{array}$$

After the student has mastered this algorithm, the next step is to show the relationship between this technique and the traditional algorithm.

$$\begin{array}{r} 1459 \\ \underline{+\ 526} \\ \textcircled{1}5 \\ 70 \\ 900 \\ \underline{1000} \\ 1985 \end{array} \longrightarrow \begin{array}{r} \overset{1}{1459} \\ \underline{+\ 526} \\ 1985 \\ \\ \\ \\ \\ \end{array}$$

- ■ Have the students calculate and write the partial sums to multidigit addition problems, then use the calculator to check their answers. Students could also write the partial sums and then use the calculator to add them.

Verbal Problems (Objective F)

When solving verbal problems, the techniques presented in Chapter 5 can be helpful. An important step is to have students restate the problem in different words, since this requires them to think about the overall situation being presented rather than applying the first technique that comes to mind. In single-digit verbal problems, the teacher may find it useful to have students represent the problem with blocks or poker chips. For example, in a problem where a farmer has cows in two pens and joins them together, the student could use plastic animals and loops of yarn, then count to find the answer.

With multidigit verbal problems involving addition, students might be asked to represent the problem with bundled sticks or on the place-value chart. Often, having the student draw a sketch of the situation can be helpful as well.

- Give road maps of the state or province you are in to groups of 3 or 4 students and ask them to find the shortest route between two cities where there are a number of possible routes. See which group can come up with the shortest route. When students come up with their route, have them mark it in red. Exchange maps and have each group check the computations of another group. In order to win, a group must correctly calculate the distance of the marked route.

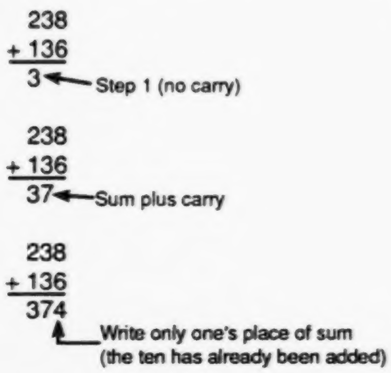
Left-to-Right Addition (Objective I)

Why do we teach children to add from right to left rather than the other way around? Perhaps because it is slightly easier—there are not quite as many steps to remember (you do not need to check ahead to see if you will need to add any regrouped values), at least when using pencil and paper. On the other hand, with the importance of estimation in the age of electronic computational devices, we may wish to teach children to add from left to right using pencil and paper. This algorithm corresponds to front-end mental addition techniques discussed below.

Figure 10.6 shows the steps in the left-to-right algorithm.

To teach this technique to students, represent written addition problems using manipulatives, such as base-ten blocks. First represent the two addends. Then, add the blocks in the largest place (in the sum "235 + 307," you would first add the hundreds) and write the sum for that place. Next, add the blocks in the next largest place, doing any regrouping necessary. If you have regrouped, it will be necessary to change the sum in the largest place. Repeat until the smallest place has been added. After two or three examples, do several problems in which you add in the largest place, but then examine the next largest place before writing the sum for the largest place to see if you will have to augment it. Repeat for the rest of the places. (See Figure 10.7.)

FIGURE 10.6
Left-to-right addition algorithm.


238
+ 136
3
Step 1 (no carry)
238
+ 136
37
Sum plus carry
238
+ 136
374
Write only one's place of sum
(the ten has already been added)

Step 1: Add digits in leftmost place.

Step 2: Check digits in next place to right. If sum in that place is greater than 10 add 1 to sum (if greater than 20 add 2, if greater than 30, add 3, etc.), if sum equals 9, check next places to right to see if there is a carry. Add carry, if necessary, and write answer.

Step 3: Repeat process in next place to right, etc.

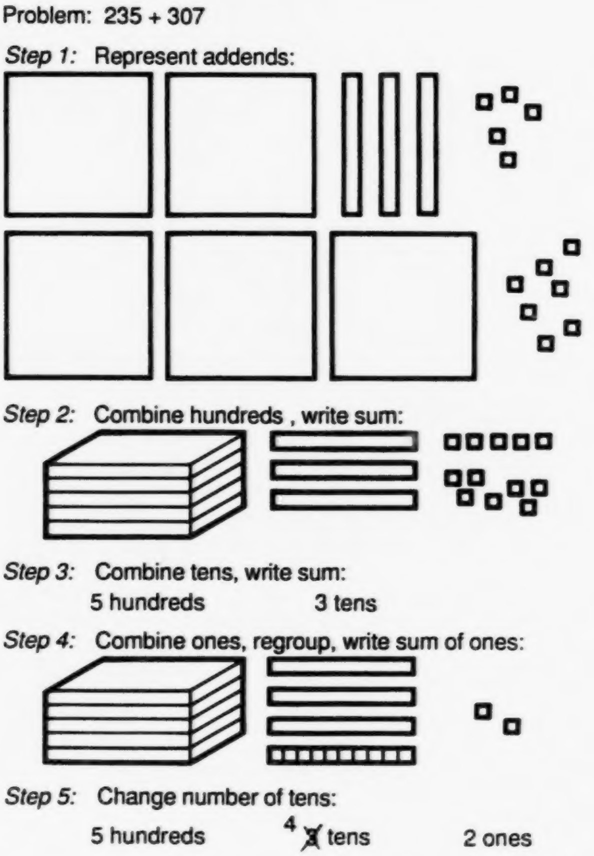

Problem: 235 + 307
Step 1: Represent addends:
Step 2: Combine hundreds , write sum:
Step 3: Combine tens, write sum:
5 hundreds
3 tens
Step 4: Combine ones, regroup, write sum of ones:
Step 5: Change number of tens:
5 hundreds
$\overset{4}{\cancel{3}}$ tens
2 ones

FIGURE 10.7
Left-to-right addition with manipulatives.

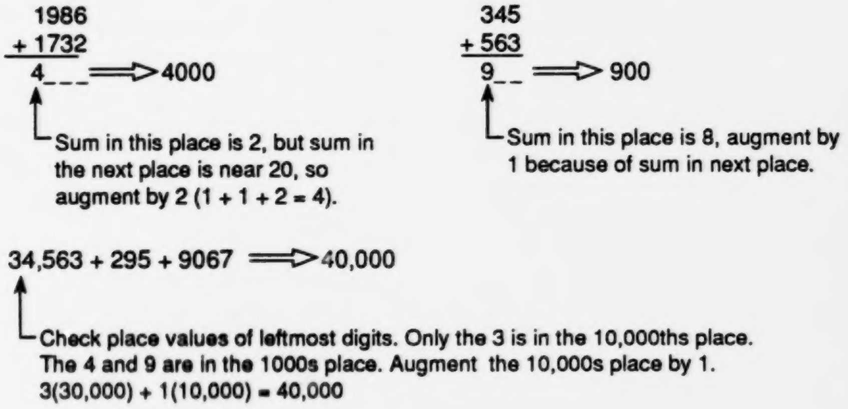
Finally, have students do many practice problems without the manipulatives. If they get stuck, ask them to imagine the manipulatives; if the difficulty persists, have them do more work with the concrete materials.

Estimating Sums (Objective L)

When estimating the sum of two or more numbers, we focus on the furthest left (or most significant) digits in the numbers. First, we must determine the place values of the numbers involved, then we carry out the addition on the appropriate values. For example, if we are estimating the sum of 34,563 and 295 and 9067 to the nearest ten thousand we only need to consider the first and last numbers.

We have various ways to estimate the sum of several numbers. The most primitive is to simply find the sum of the left-most places (assuming that they have the same place value) and insert zeros in all places to the right. The estimate for the sum of 345 and 563 using this technique would be 800. Of course, this method will not produce very accurate results if the left-most digits in the numbers are small and the next digits are large (consider "1986 + 1732"—the estimated sum using this method is about half the actual sum).

One method for obtaining more accurate results is to adjust the answer (compensate) by examining the second place from the left. If the sum in that place is around 10, we add one to the estimated sum in the left-most place. If the sum approaches 20, we add 2 to the left-most place. This process is outlined in Figure 10.8.



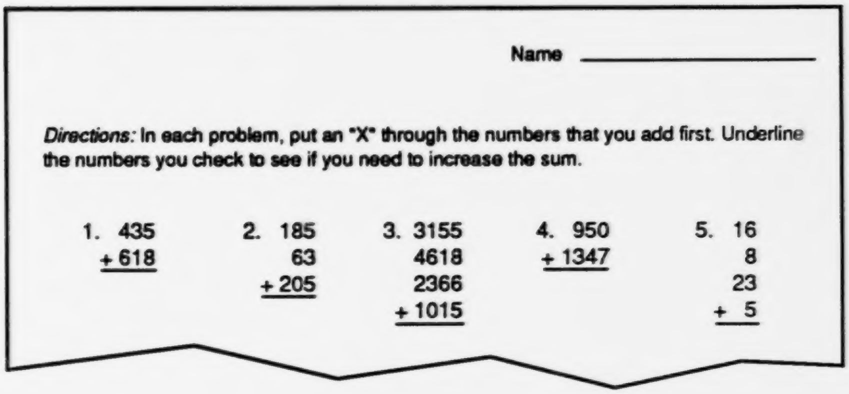
Sum in this place is 2, but sum in the next place is near 20, so augment by 2 (1 + 1 + 2 = 4).

Sum in this place is 8, augment by 1 because of sum in next place.

Check place values of leftmost digits. Only the 3 is in the 10,000ths place. The 4 and 9 are in the 1000s place. Augment the 10,000s place by 1. 3(30,000) + 1(10,000) = 40,000

FIGURE 10.8
Examples of addition estimation algorithm.

■ When teaching children to use this technique for estimating the sums of numbers, start by giving them problems in the traditional, vertical format with the places in the numbers lined up. After an appropriate discussion, give them a worksheet in which they simply circle or mark those digits that should be considered when estimating the sum (see Figure 10.9).



Name ____________________

Directions: In each problem, put an "X" through the numbers that you add first. Underline the numbers you check to see if you need to increase the sum.

1.	2.	3.	4.	5.
435	185	3155	950	16
+ 618	63	4618	+ 1347	8
	+ 205	2366		23
		+ 1015		+ 5

FIGURE 10.9
A sample worksheet involving estimation of sums.

Next, include problems in the horizontal form:

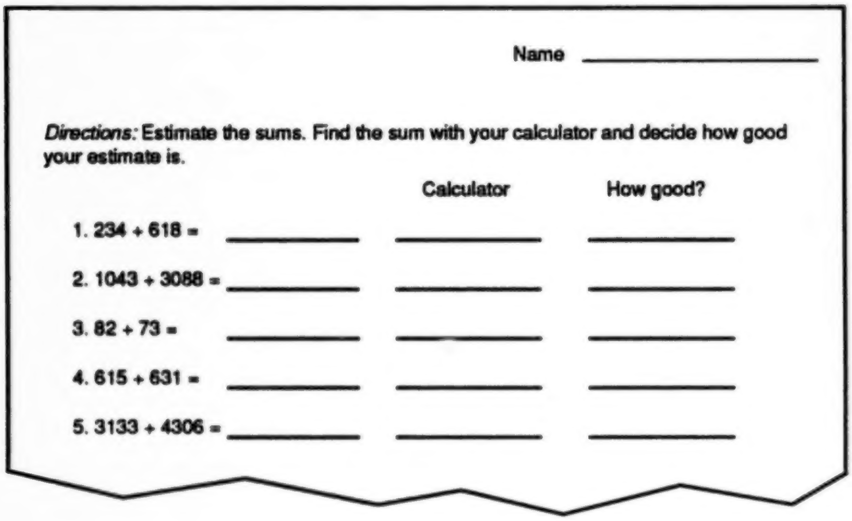
EXAMPLE: Put an "X" through the numbers we should not use in estimating the sum.

1. 2861 + 43 + 4512 + 9 = ?
2. 23,456 + 30,992 + 1207 = ?

Also, do similar activities with verbally-stated addition problems. Of course, the addends in the problems provided should be at the level of the students.

■ Have them use the techniques discussed above to estimate the sums in problems such as those above. After each problem, they should use calculators to determine the exact answer and then evaluate the accuracy of their estimate. Start with problems where no compensation is necessary and all addends have the same number of places. In later worksheets (after appropriate instruction), include problems where compensation is necessary and where the addends have different numbers of places. (See Figures 10.10 and 10.11.)

FIGURE 10.10
An early worksheet for addition estimation.

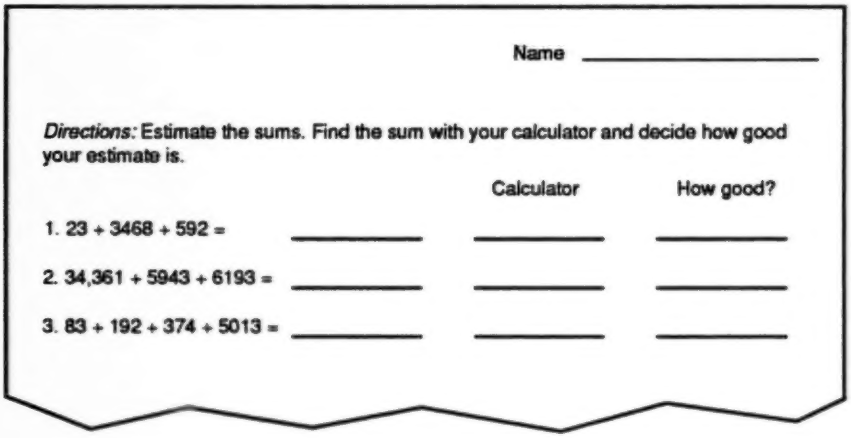


Name ____________________

Directions: Estimate the sums. Find the sum with your calculator and decide how good your estimate is.

		Calculator	How good?
1. 234 + 618 =	________	________	________
2. 1043 + 3088 =	________	________	________
3. 82 + 73 =	________	________	________
4. 615 + 631 =	________	________	________
5. 3133 + 4306 =	________	________	________

FIGURE 10.11
A more advanced worksheet for addition estimation.



Name ____________________

Directions: Estimate the sums. Find the sum with your calculator and decide how good your estimate is.

		Calculator	How good?
1. 23 + 3468 + 592 =	________	________	________
2. 34,361 + 5943 + 6193 =	________	________	________
3. 83 + 192 + 374 + 5013 =	________	________	________

A more sophisticated method for obtaining estimated sums is to round the numbers before adding the front digits. Students need to thoroughly understand the numeration concepts discussed in the previous chapter to use this strategy. As with the method above, before you round and add, you must first decide which of the addends have enough places to contribute to the sum.

Next you mentally round each of the appropriate addends and then add the leftmost one or two digits. For example:

PROBLEM: 386 + 1596 + 2413 = _____

1. Round first two to nearest hundred and add (mentally):

   400 + 1600 = 2000

2. Round next number to nearest hundred and add:

   2000 + 2400 = 4400

Activities for teaching this technique should be similar to those discussed for teaching estimating.

A final word about estimation and mental arithmetic: do not delay instruction in this important skill until the middle or upper elementary grades. Begin in the first grade—as soon as children begin working with two-digit, symbolic addition. In addition to the formal estimation and mental computation activities discussed here, encourage and even require children to estimate answers first whenever they compute answers using pencil and paper or calculators, or when they do problem-solving activities, so that it becomes second nature to them.

SUBTRACTION OF WHOLE NUMBERS

The idea of subtraction is more complex than addition because there are four different physical situations that suggest subtraction:

Set Separation: Professor Barnes is lecturing a class of 45 students about using games in teaching mathematics. He makes the statement, "Blondes have more fun," whereupon the 11 blondes in the class get up and leave to go have a party. How many people stay in the class to hear the end of the lecture?

Set Comparison: John has collected 8 butterflies, and Raul has collected 3 butterflies. How many more does John have than Raul? (See Figure 10.12.)

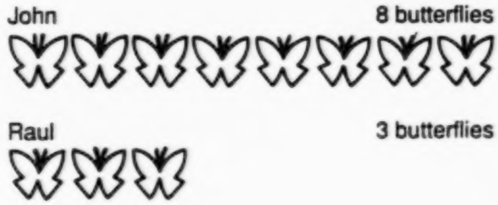
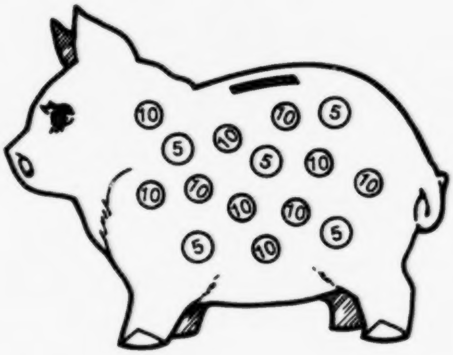


FIGURE 10.12
A set comparison example.

Missing Addend: Stephanie wants to buy a bicycle that costs $98. She has saved $42 so far from her paper route. How much more does she need to earn?

Subset: Ralph has 15 coins in his piggy bank. The coins are either nickels or dimes. If 5 of the coins are nickels, how many are dimes? (See Figure 10.13.)

FIGURE 10.13
A subset meaning of subtraction.



Behaviorial Objectives for Subtraction of Whole Numbers

A. Use manipulative materials to solve a verbally-presented subtraction problem involving set separation (minuend < 19 and subtrahend < 10).

B. Use manipulative materials to solve a verbally-presented subtraction problem involving set comparison (minuend < 19 and subtrahend < 10).

C. Use manipulative materials to solve a verbally-presented subtraction problem involving a missing addend (minuend < 19 and subtrahend < 10).

D. Use manipulative materials to solve a verbally-presented subtraction problem involving subsets (minuend < 19 and subtrahend < 10).

E. Solve addition problems with a missing addend (addends < 10).

F. When shown a written subtraction fact (minuend < 19 and subtrahend < 10) in vertical or number-sentence form, use materials, pictures, or counting techniques to represent the problem and find the correct answer.

G. State subtraction facts given in written form (minuend < 19 and subtrahend < 10) without using manipulative materials or counting techniques.

H. Represent multidigit subtraction problems (take-away model) with and without regrouping (using bundled sticks or base-ten blocks) and state the answer.

I. Represent multidigit subtraction problems (take-away model) with and without regrouping on a place-value chart or abacus and state the answer.

J. Subtract a number from a larger number, both without and then with regrouping, using pencil and paper (using an alternative or the standard algorithm).

K. Subtract a number from another number using left to right subtraction.

L. Estimate the difference of two numbers using front-end subtraction.

M. Solve verbal and applied problems involving subtraction.

N. Use a calculator to subtract a value from another.

O. Use the PRINT statement in Logo or BASIC to subtract one value from another.

TEACHING SUBTRACTION

Activities with Single Objects for Teaching Subtraction (Objectives A–F)

Students need to experience activities with set separation, set comparison, missing addends, and subsets involving physical objects in order to lead them to understand the various meanings of subtraction. Subset activities may be postponed until later grades since this concept poses difficulty for some children.

Note: do not attempt to teach more than one of the meanings of subtraction in one sitting. Students should be presented with one of the meanings then given ample opportunity to experience it through appropriate manipulations and other experiences before another meaning is presented. Indeed, you may wish to present only the take-away meaning initially, then wait a few weeks or more to present the other meanings.

■ Demonstrate the *take-away model* using single objects on the overhead or demonstration-sized models. Ask the students to make a set containing a given number of objects (less than 19). Then ask them to remove a given number of objects from the original set (differences should be less than 10). Ask them to count to find how many are left.

■ Place 25 single objects (e.g., dried garbanzo beans) in front of the students. Use the beans to demonstrate several subtraction operations.

For example, say "I am going to show you twelve minus (take-away) seven." Place 12 beans in a group. Say, "We have twelve here. Now I am going to take away seven of them." Remove seven. Pointing at the remaining beans, say "Now, I have one, two, three, four, five left."

After several examples, ask students to show you several subtraction problems: for example, "Show me eleven minus eight with the beans." After the children manipulate the beans, ask "What is the answer?"

■ Demonstrate the *set comparison* meaning of subtraction with single objects. Make two sets of objects (minuend < 19, both the subtrahend and difference $<$ 10). Demonstrate how to find out how many more there are in one set than in the other. For example, use one-to-one matching, then count the number of unmatched objects. Include problems where the difference is zero. After several examples, ask the students to show you several subtraction problems involving *differences*. For example, "Show me the difference between five and nine with the beans." Then ask, "What is the answer?"

■ Demonstrate the *missing addends* meaning of subtraction using objects. Make a set with a given number of objects (< 10). Show how you would find how many more it would take to make a larger set with a given number of objects. For example, say, "I have five beans." Make a set with five beans. "How many more beans do I need to have 12 in the set?" Then add beans, one at a time while counting, until you have 12 in the set:

● ● ● ● ● How many more to have 12?

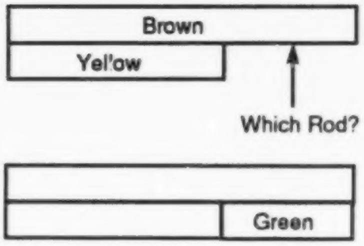
■ After several examples, ask the student to show you how to find answers to missing addend problems. For example, instruct them to use the beans to find out what number plus eight is fifteen. Do several examples using different single-digit sums.

- Use poker chips to demonstrate the *subset* meaning for subtraction. For example, make a set of 13 chips with 7 red and 6 white chips and say, "If I have 13 chips altogether and 7 of the chips are red, how many are white?" State that this shows 13 minus 7. Do several such examples.

- Ask the student to use the chips to show you other subtraction problems, then to find the answers. For example, tell them "Show me fifteen minus nine with the colored chips. What is the answer?"

- Cuisinaire rods lend themselves to the comparison idea of subtraction. To model 8 − 5, the child places a brown and a yellow rod side by side and finds the rod which fits in the empty space—the rod that tells you how much longer the eight rod is than the five (see Figure 10.14). The rods may be used to illustrate the missing addend idea also. Ask students "What rod would I place beside the yellow rod to make the length of a brown rod?

FIGURE 10.14
Cuisenaire rod subtraction.

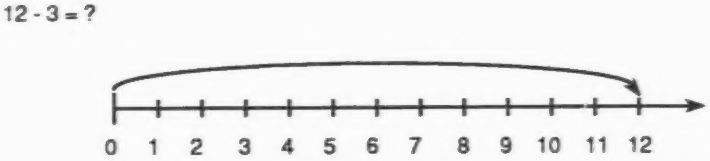


5 + ______ = 8 or 8 − 5 = ______

What is *missing?*"*

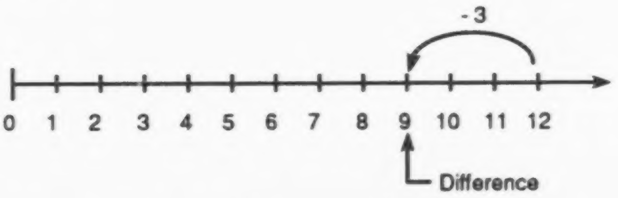
- Another useful manipulative is the *number line*. The take-away or set-comparison meanings of subtraction may be modeled by first drawing an arrow to the right from zero to the minuend (Figure 10.15).

FIGURE 10.15
Number line subtraction.



Next, draw an arrow from the endpoint of the minuend to the left for the number of units in the subtrahend. The new endpoint of the arrow is the difference (Figure 10.16).

FIGURE 10.16
Number line subtraction continued.



To model the missing addend meaning of subtraction, first draw an arrow from zero to the known addend (Figure 10.17).

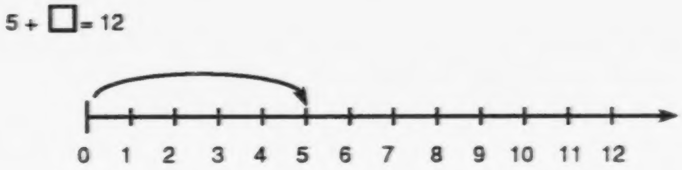


FIGURE 10.17
Number line subtraction: Missing addend.

Next, note where the sum is located on the number line and count the number of units between the first addend and the sum* (Figure 10.18).

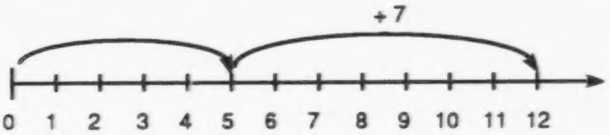
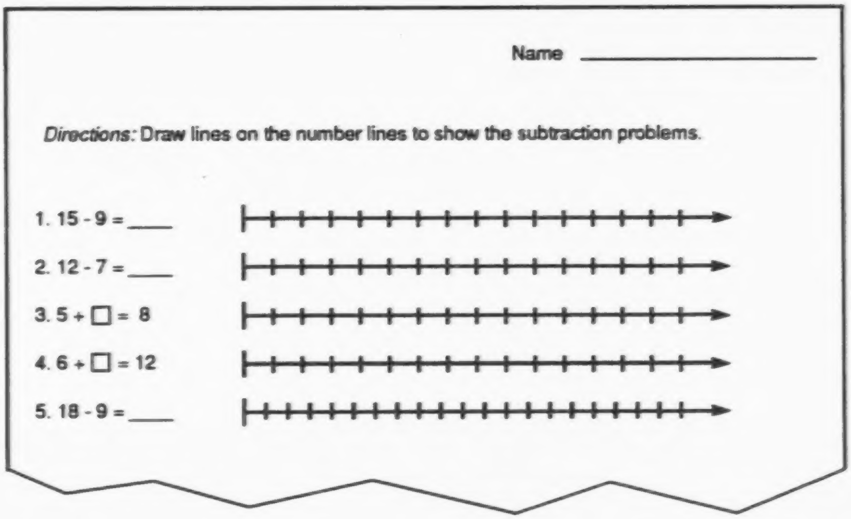


FIGURE 10.18
Number line subtraction continued.

■ After appropriate demonstrations for your students, reproduce worksheets with several blank number lines and give them written subtraction problems to solve with the number lines. Include subtraction in both the standard form and the missing addend form. (See Figure 10.19.)



Name ____________

Directions: Draw lines on the number lines to show the subtraction problems.

1. 15 - 9 = ____
2. 12 - 7 = ____
3. 5 + □ = 8
4. 6 + □ = 12
5. 18 - 9 = ____

FIGURE 10.19
Number line subtraction worksheet.

Relationship Between Addition and Subtraction (Objective E)

■ Demonstrate the relationship between addition and subtraction problems. Have students rewrite addition problems with a missing addend as subtraction problems.

EXAMPLE: 1. 5 + ____ = 9 9 − 5 = ____
2. 7 + ____ = 12 12 − 7 = ____

Most teachers do not spend as much time drilling subtraction facts as they do addition facts. If the relationship between addition and subtraction is made clear, students will be able to determine answers to subtraction problems relatively quickly.

■ The unknown subset idea may be illustrated by a simple guessing game. Say, "I have eight marbles. Count them. I'm going to put them behind my back and put some marbles in one hand and some in the other." Take your hands from behind your back and open one, showing the number of marbles in that hand (5). "How many were in the set that I had in the beginning? How many are in the set in my open hand? Guess how many are in the set in my closed hand?"*

■ Have children use the addition table to find missing addends. In the problem 11 − 3 = ____, for example, ask the students to find the 3 on the outside column then follow the row until they reach 11. Move up the column to find 8.* (See Figure 10.20.)

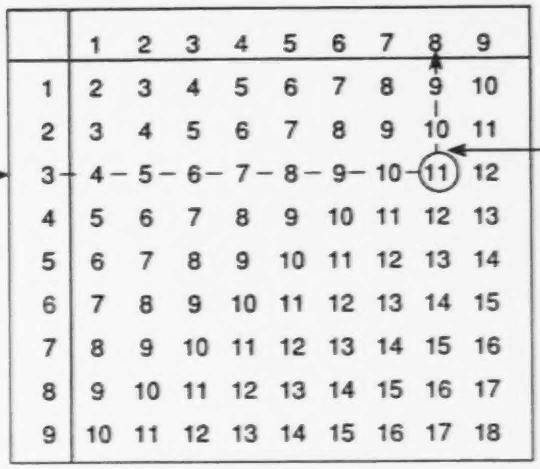


FIGURE 10.20
Subtraction with an addition table.

Look for 11 in this row.

Move up this column to find difference.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

■ Practice subtraction fact in ways similar to the methods introduced earlier for practicing addition. Flashcards, games, and timed drills can all be used successfully.

Learning Subtraction Facts (Objective G)

A crucial step in teaching basic subtraction facts to students is associating symbols with manipulative activities such as those described above. Another important step is leading the student to discover the relationship between single-digit addition and basic subtraction facts. (By basic subtraction facts we mean problems involving differences and subtrahends less than ten.)

---

*Activity by Jewell Garner.

- Repeat the above activities for the take-away model and write a number sentence that corresponds to the manipulations. For example, make a set containing thirteen objects and write 13; remove seven objects and write − 7 = . State the answer and write 6. Go through similar sequences for the other meanings of subtraction.

- Relate manipulations to the vertical form of subtraction problems. For example, relate set comparison to

$$\begin{array}{r} 7 \\ \underline{-6} \end{array}$$

- To help the student understand the grouping concepts involved in subtraction problems, solve basic subtraction fact problems using bundled sticks. In the problem 12 − 9, for example, represent the 12 with a bundle of ten and 2 units. The next step is to regroup (unbundle the ten). Now the subtraction can take place. Although it may seem that regrouping is not important when working with basic subtraction facts, these activities pave the way for later work with multidigit subtraction.

- Show the student written subtraction problems in number sentence and in vertical form. Have her solve them using the bundled sticks. (Figure 10.21.)

FIGURE 10.21
Open number sentences and vertical subtraction.

$9 - 5 = \square$

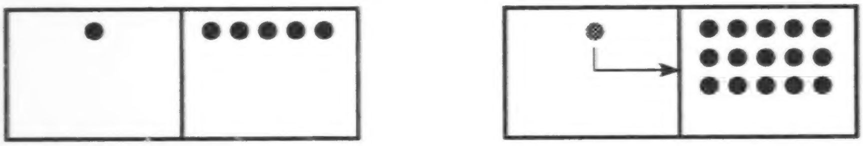
$7 + \square = 15$

$$\begin{array}{r} 9 \\ -3 \\ \hline \square \end{array} \qquad \begin{array}{r} 5 \\ +\square \\ \hline 13 \end{array} \qquad \begin{array}{r} \square \\ +8 \\ \hline 11 \end{array}$$

- Solve problems similar to those above using the place-value chart. In the problem 15 − 9, the student would place 1 marker in the ten's place and 5 in the one's place. To subtract 9 ones, he would exchange the ten for 10 ones. After 9 units had been taken away, the answer would be recorded. (Figure 10.22.)

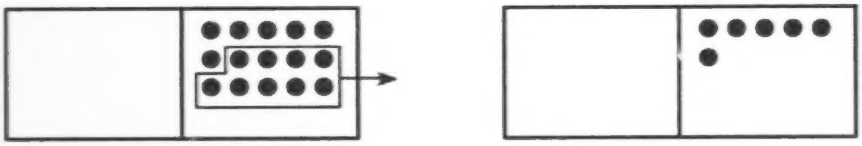
FIGURE 10.22
Place value chart subtraction.

15 - 9 =



1. Represent minuend

2. Regroup



3. Subtract

4. Write answer 6

SUBTRACTION PRINCIPLES

Perhaps one of the most striking differences between subtraction and addition is the fact that two important principles—commutativity and associativity—do not apply to subtraction. For example,

$$25 - 47 \neq 47 - 25$$

and

$$10 - (5 - 7) \neq (10 - 5) - 7$$

■ If you wish to introduce integers and appropriate calculator techniques (see Chapter 17), you may wish to have your children use their calculators to experiment to see if the commutative and associative properties work for subtraction. After they have simplified the right and left side of a number of equations similar to those above, discuss their findings.

There are a number of other generalizations relevant to subtraction that children should be led to discover or be taught in other ways.

(1) The Zero Property for Subtraction

This general principle is that when *zero* is subtracted from a number, the result is the original number ($5 - 0 = 5$). Note that, since subtraction is *not* commutative, it is *not* true that when you subtract any number from zero you get that number ($0 - 5 \neq 5$).

This property may seem trivial to an adult with many long years of mathematical study and experience, and perhaps unworthy of our time and effort in the classroom. However, the importance of understanding this property (and others, such as the lack of commutativity in subtraction) is illustrated in these common errors made by children who have not mastered subtraction:

$$\begin{array}{r} 240 \\ -\ 28 \\ \hline 228 \end{array} \qquad \begin{array}{r} 828 \\ -200 \\ \hline 600 \end{array}$$

Children making the first error may be unsure about regrouping, and as a result, may do the "next best thing" and erroneously apply the commutative property to subtraction, thinking that if eight minus zero is eight, then zero minus eight must also be eight. The second error often occurs after children have begun to study multiplication. After learning that anything times zero is zero, they will sometimes apply this principle (the *zero* property of multiplication) to other operations, especially subtraction.

■ To teach the zero property of subtraction, ask the children to make sets of a given size with common objects, then ask them to take away zero objects. After a couple of examples with the group, you may wish to give students a worksheet with problems like the following and ask them to use counting objects to solve them.

8 − 0 = ______
5 − 0 = ______
10 − 0 = ______

You may also wish to have them attempt to represent a couple of problems like the following using manipulatives to illustrate that subtraction is not commutative, and as a lead into regrouping:

0 − 8 = _____
0 − 5 = _____

Similar activities may be done for the other meanings of subtraction. For example, if you have eight marbles, how many more do you need to get to have eight?

Finally, ask them to state a generalization about subtracting zero.

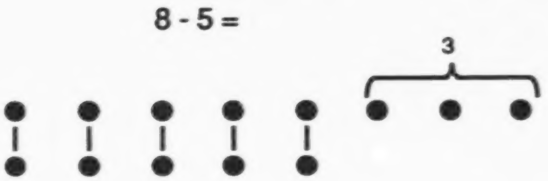
(2) Compensation

This principle states that if you add the same quantity to both the minuend and subtrahend the difference is unchanged. The problem 18 − 3 produces the same results or difference as 19 − 4 and 20 − 5. In general,

$$a - b = (a + c) - (b + c).$$

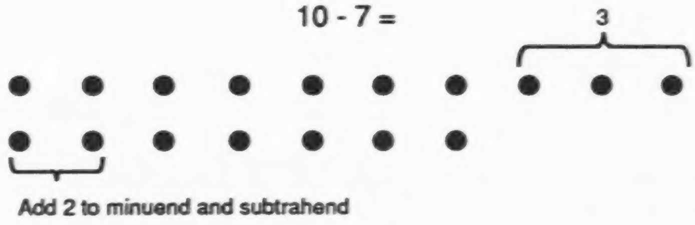
Meaning for the compensation principle is best developed using set comparison. For example, if students represent 8 − 5 using two sets, they find that there are three unmatched objects in the first set (Figure 10.23).

FIGURE 10.23
Compensation with objects.



If we add two objects to each set, we now have the problem 10 − 7 and there are still three unmatched objects (Figure 10.24).

FIGURE 10.24
Compensation with objects continued.



Thus, although the problem has been transformed to a new one, the answer remains the same.

This principle is useful when solving certain types of problems. For example, if the student has difficulty with 27 − 8, compensate the minuend and subtrahend by 2 and you have 29 − 10, an easier problem to solve.

Bundled-Stick Activities (Objective H)

A common mistake made by students learning to subtract is to apply the aphorism "take the little number from the big number," regardless of whether or not the larger digit is located in the subtrahend. Another common mistake is to record a zero in the answer when a zero appears in the minuend.

EXAMPLES:

$$\begin{array}{r} 342 \\ -\ 15 \\ \hline 333 \end{array}\qquad \begin{array}{r} 507 \\ -\ 26 \\ \hline 501 \end{array}$$

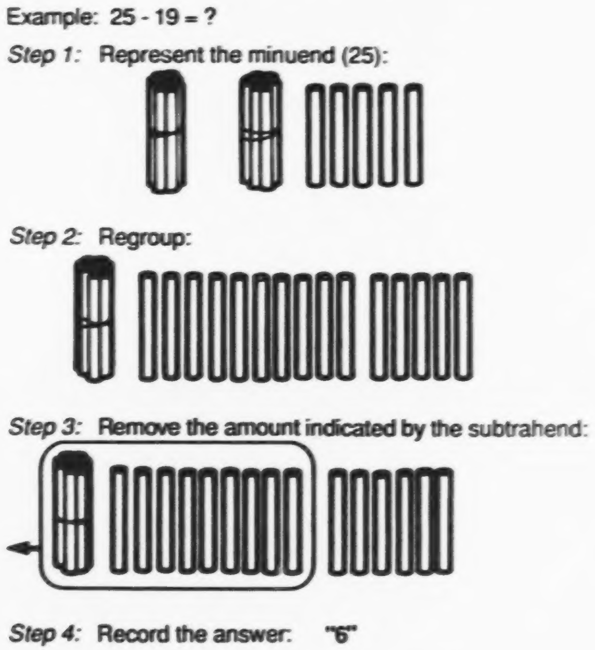
Such errors often indicate students are simply carrying out a rote sequence of steps for subtracting rather than attacking the problems with understanding. If students have appropriate experience with manipulative materials, they are much less likely to make such mistakes.

Concrete activities for developing understanding of the subtraction algorithm should be limited to the take-away model because the symbolic manipulations of the standard algorithm and the left-to-right algorithm parallel the steps we use in the take-away model with manipulatives.

■ With bundled sticks the sequence of manipulations is (1) representing the minuend; (2) regrouping if necessary; (3) removing the appropriate number of sticks; and (4) recording the answer. Note that these manipulations model the take-away meaning of subtraction. This corresponds with and leads up to the subtraction algorithm usually taught in schools.

EXAMPLE: 25 − 19 = ____ (Figure 10.25)

FIGURE 10.25
Bundled stick subtraction.



Example: 25 - 19 = ?

Step 1: Represent the minuend (25):

Step 2: Regroup:

Step 3: Remove the amount indicated by the subtrahend:

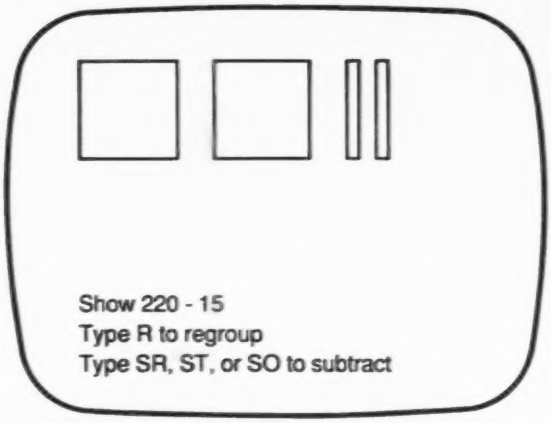
Step 4: Record the answer: "6"

Beginning activities may involve stating problems verbally or presenting them in written form, having the student solve the problem with the sticks, then having him say or write the answer.

- Present written problems (both as number sentences and in vertical format) to the student, have her use the sticks to solve the problem, then have her record the answer.

- The disk for this textbook contains a BASIC program called "HTO-SUBTRACT," designed to help students understand multidigit subtraction through graphics manipulations. To use the program, boot your system with the SuperPilot lesson diskette and select HTO-SUBTRACT. The program places subtraction problems on the screen and asks students to represent the problems with pictures of "flats," "longs," and "units" (hundreds, tens, and ones) by pushing the H, T, and O buttons. Each time the student pushes the O (ONES) button, for example, a small square or unit appears on the screen. When the screen display is to the student's liking, he pushes the K (for "OK") button.

  If the minuend has not been correctly represented, the program asks the student to keep trying. If he wishes to regroup, he types "R." The computer then asks if he wishes to regroup hundreds (to tens) or tens (to ones). After regrouping, the student subtracts hundreds, tens, or ones by typing SH, ST, or SO each time he wishes to remove one of these. When he is finished subtracting, he types "A" to indicate that the screen display is the answer (difference) and the computer tells him if his answer is correct, clears the screen, and represents another problem. A sample of the screen display is shown in Figure 10.26.



Show 220 - 15
Type R to regroup
Type SR, ST, or SO to subtract

FIGURE 10.26
A screen display of HTO SUBTRACT program.

Place-Value Chart Activities (Objective I)

Place-value chart activities with subtraction are very helpful in developing understanding of the standard subtraction algorithm because the manipulations duplicate the computational steps involved in regrouping (borrowing). Figure 10.27 is a demonstration of the sequence of steps for three-digit minus three-digit subtraction with regrouping.

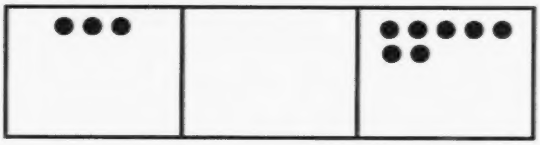
PROBLEM: 307 − 128 = ______

FIGURE 10.27
Place-value chart subtraction.

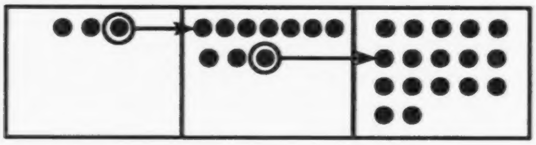
Problem: 307 - 128 = ____

Solution:

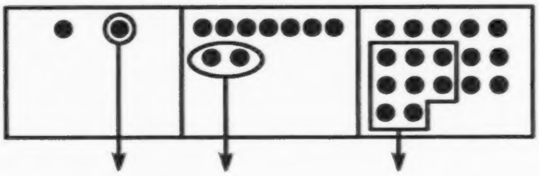
Step 1: Represent minuend:



Step 2: Regroup so that 2 tens and 8 ones can be subtracted:



Step 3: Remove amount indicated by subtrahend:



Step 4: Write answer: "179"

Students should be given a variety of written problems and asked to do place-value chart manipulations to determine answers, then write them down.

Point out the relationship between the manipulations and the computational steps in the standard algorithm.

EXAMPLE 1: Solve the following using the place-value chart. (Write your answers in the blanks provided.)

45 − 19 = ____________________

80 − 37 = ____________________

EXAMPLE 2: Solve the following using the place-value chart. (Write your answer in the space under the problem.)

$$\begin{array}{r} 3045 \\ -\ 147 \\ \hline \end{array} \qquad \begin{array}{r} 5000 \\ -1507 \\ \hline \end{array}$$

The Standard Algorithm (Objective J)

After students are able to do multidigit subtraction with physical materials, they can use many activities at the symbolic level that promote understanding of and competence with subtraction. A sequence of instruction for developing meaning for subtraction can involve many of the same techniques discussed in the sections on addition and place value. One useful method is expanded notation. The essential step in this method is regrouping (e.g., expressing a number like 50 + 7 as 40 + 17). See Chapter 7 for further discussion.

PROBLEM: $57 - 29 = ?$

STEP 1: Express in expanded notation:

$$(50 + 7) - (20 + 9)$$

STEP 2: Regroup (noting that $7 - 9$ is a negative number):

$$(40 + 17) - (20 + 9)$$

STEP 3: Simplify (remember you are subtracting *both* 20 and 9):

$$40 + 17 - 20 - 9$$

STEP 4: Change order of numbers, grouping tens with tens and ones with ones:

$$(40 - 20) + (17 - 9)$$

(Note: we are not violating the noncommutative property here. Rather, we are applying the commutative property for addition of integers: $(+17) + (-20) = (-20) + (+17)$

STEP 5: Subtract tens and ones:

$$20 + 8 = 28$$

It would not be wise to ask children to produce a sequence like this by themselves, but many will benefit by following the teacher through several such step-by-step demonstrations.

The following sequence accomplishes the same goals as the above sequence. However, since it is in the vertical form (more commonly used in subtraction problems), the student has fewer parentheses to deal with, and does not need to worry about dealing with the commutative property. It should therefore be easier for him to follow. Note the accompanying base-ten block manipulations (Figure 10.28).

Rather than having the student reproduce the entire sequence, write out the sequence in nearly complete form and have him complete the blanks.

EXAMPLE:

$$\begin{array}{r} 400 \\ \underline{-\ 45} \end{array} \qquad \begin{array}{r} 400 + \ 0 + 0 \\ \underline{\qquad -\ 40 - \_\_} \end{array} \qquad \begin{array}{r} 300 + \_\_ + 0 \\ \underline{\qquad -\ 40 - 5} \end{array} \qquad \begin{array}{r} 300 + 90 + \_\_ \\ \underline{\qquad -\ 40 - 5} \end{array}$$

■ Another method is to use a place-value chart (using written numerals) to distinguish between 10 in the tens' place and 10 ones in the ones' place, and so on. The place-value chart with written numerals closely approximates the steps used in solving similar problems with a manipulative place-value chart.

FIGURE 10.28
Meaning for multidigit subtraction.

Problem:

$$\begin{array}{r} 223 \\ \underline{-\ 17} \end{array}$$

Step 1: Express in expanded notation:

$$\begin{array}{r} 200 + 20 + 3 \\ \underline{-(\quad 10 + 7)} \end{array}$$

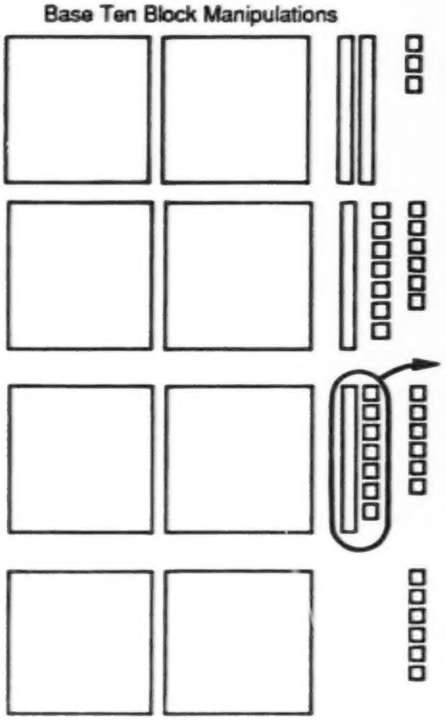
Step 2: Regroup:

$$\begin{array}{r} 200 + 10 + 13 \\ \underline{-(\quad 10 + 7)} \end{array}$$

Step 3: Subtract:

$$\begin{array}{r} 200 + 10 + 13 \\ \underline{-10 - \ \ 7} \\ 200 + 0 + 6 \end{array}$$

Answer: 206



PROBLEM:

$$\begin{array}{r} 116 \\ \underline{-\ 89} \end{array}$$

(See Figure 10.29 on the following page.)

When teaching the steps in the standard algorithm for subtraction, present the steps while manipulating bundled sticks or a place-value chart. The simple instructional sequence in Figure 10.30 may be helpful.

Try to wean children away from the manipulatives, but keep them available in case difficulties or misunderstandings arise.

Equal Additions Algorithm

An alternative to the standard algorithm (sometimes called the *decomposition* algorithm) is the *equal additions* method for subtraction. Although the reasons why this algorithm works are more obscure than for the decomposition algorithm, the steps are easier for children to learn and may be of use with children who cannot seem to master the steps in the standard algorithm.

The equal additions algorithm is based on the compensation principle presented above. The steps of this technique are no different from the standard algorithm, unless a digit in the subtrahend is greater than the digit in the same place in the minuend.

$$\begin{array}{r} 4\ 5 \\ \underline{-1\ 7} \end{array}$$

FIGURE 10.29
Symbolic place-value chart subtraction.

Problem: $\begin{array}{r} 116 \\ -89 \\ \hline \end{array}$

Solution:

Step 1: Write problem on place-value chart:

Hundred	Ten	One
1	1	6
-	8	9

Step 2: Regroup:

Hundred	Ten	One
1	0	16
-	8	9

Hundred	Ten	One
0	10	16
-	8	9

Step 3: Subtract and write answer:

Hundred	Ten	One
	10	16
	- 8	9
	2	7

If that is the case, instead of regrouping you add equal values to that place in the minuend, and in the next place to the left in the subtrahend. In the problem above, for example, you add 10 ones to the minuend, making it 4 tens and 15 ones. You also add 10 to the subtrahend, but in the tens' place.

$$\begin{array}{r} 4\;5 \\ -1\;7 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{r} 4\;{}^{1}5 \\ {}^{2}\cancel{1}\;7 \\ \hline 28 \end{array}$$

The result is a transformed problem which produces the same result as the original:

$$(45 - 17) = (45 + 10) - (17 + 10)$$

Figure 10.31 shows two further examples of the use of this algorithm.

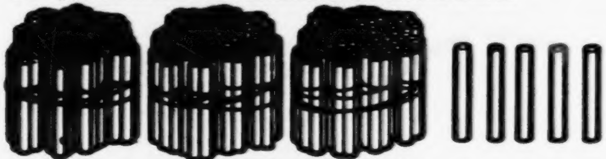
■ To teach this technique, start with manipulative activities. First, review the compensation principle for subtraction using problems like 12 − 6 and 10 − 5. Next, represent the minuend in problems like 45 − 17 and discuss the problem of subtracting in the ones' place. Next, transform the problem into one that requires

FIGURE 10.30
Developing the standard algorithm.

Teacher: "We are going to work the problem 305 minus 138."

$$\begin{array}{r} 305 \\ -\ 138 \\ \hline \end{array}$$

"First, let's make three-hundred five with the sticks."



"Now, can we take away 8 ones?" (No)

"So, what do we do so that we can?" (Regroup)

"That's right. We take one bundle of one hundred and take the rubber band off. So we will cross out the 3 and mark a 2 to help us remember that we only have two left. And we put a 1 here to show that we now have 10 tens."

$$\begin{array}{r} \overset{2}{\cancel{3}}\overset{1}{}05 \\ -\ 138 \\ \hline \end{array}$$

"What do we do now?" (Regroup one of the tens)

"Right. How many tens do we have left?" (nine)

"So, what should we do to show that?" (Mark out the 10, write 9)

$$\begin{array}{rrrr} & \overset{2}{\cancel{3}} & \overset{9}{\cancel{10}} & \overset{1}{5} \\ - & 1 & 3 & 8 \\ \hline \end{array}$$

"How many ones do we have now?" (fifteen)

"How can we show that?" (write 15)

"Now can we subtract?" (yes)

"How many ones do we have? How many tens? How many hundreds?" (Write numbers below)

FIGURE 10.31
Equal additions algorithm examples.

$$\begin{array}{lrrrrr} & & \textit{Step 1:} & \textit{Step 2:} & \textit{Step 3:} & \textit{Step 4:} \\ 1. & 3461 & 346{}^{1}1 & 346{}^{1}1 & 34{}^{1}6{}^{1}1 & 34{}^{1}6{}^{1}1 \\ & \underline{-\ 1628} & \underline{-\ 16{}^{3}\cancel{2}8} & \underline{-\ 16{}^{3}\cancel{2}8} & \underline{-{}^{2}\cancel{1}6{}^{3}\cancel{2}8} & \underline{-{}^{2}\cancel{1}6{}^{3}\cancel{2}8} \\ & & & 33 & 33 & 1833 \\ 2. & 500 & 50{}^{1}0 & 50{}^{1}0 & 5{}^{1}0{}^{1}0 & 5{}^{1}0{}^{1}0 \\ & \underline{-\ 46} & \underline{-\ {}^{5}\cancel{4}6} & \underline{-\ {}^{5}\cancel{4}6} & \underline{-1{}^{5}\cancel{4}6} & \underline{-1{}^{5}\cancel{4}6} \\ & & & 4 & 4 & 454 \end{array}$$

no borrowing (e.g., add 3 to the minuend and subtrahend in the problem 45 − 17; the resulting problem is 48 − 20). Discuss the problem of knowing how many to add to the minuend and subtrahend. Show students a method that will always work—adding ten units to the minuend and another ten to the subtrahend. Finally, give students similar problems to practice with.

Left-to-Right Subtraction (Objective K)

When we find differences mentally or when we estimate the difference of two numbers, we tend to concentrate on the most significant (or left-most) digits in the problem. On the other hand, the usual way of subtracting using pencil and paper is to start with the least significant digits and move to the left. In order to prepare children for mental subtraction, we may wish to teach them left-to-right subtraction using pencil and paper. The steps of this technique follow:

$$\begin{array}{r} 327 \\ -\,162 \\ \hline \end{array}$$

1. Write the minuend and subtrahend in the usual manner, lining up the places.

$$\begin{array}{r} 3\boxed{2}7 \\ -\,1\boxed{6}2 \\ \hline \end{array}$$

The "2" is less than the "6" so reduce the difference in hundreds place by 1.

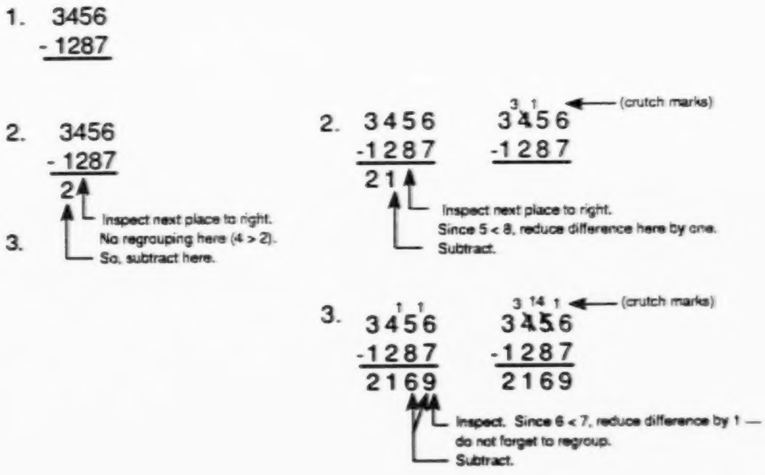
2. Inspect the place immediately to the right of where the subtraction will begin (i.e., the left-most place). If the digit in the minuend is *greater* than the digit in the subtrahend, no regrouping is needed. If the digits are the *same* size, inspect the digits two places to the right to see if regrouping will be needed. If so, reduce the difference by one. If the digit in the minuend is *less than* the digit in the subtrahend, reduce the difference by one.

$$\begin{array}{r} 327 \\ -\,162 \\ \hline 1 \end{array}$$

3. Subtract, then move to the next place to the right. Repeat starting at #2.

See Figure 10.32 for another example of this method.

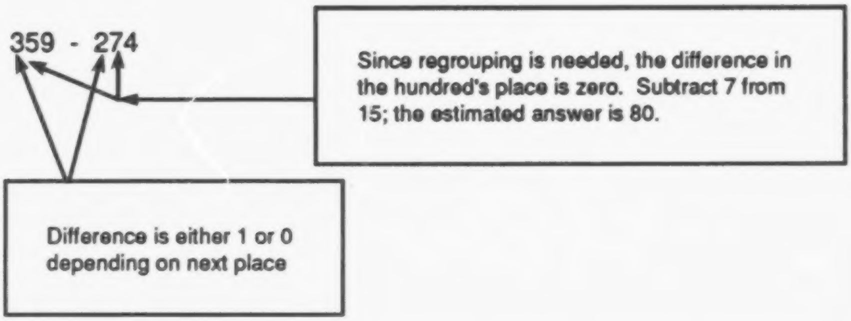
FIGURE 10.32
Left-to-right subtraction.


1. 3456
- 1287
2. 3456
- 1287
2
3.
Inspect next place to right.
No regrouping here (4 > 2).
So, subtract here.
2. 3456
-1287
21
3 1
3456
-1287
(crutch marks)
Inspect next place to right.
Since 5 < 8, reduce difference here by one.
Subtract.
3. 1 1
3456
-1287
2169
3 14 1
3456
-1287
2169
(crutch marks)
Inspect. Since 6 < 7, reduce difference by 1 —
do not forget to regroup.
Subtract.

Front-End Subtraction (Objective L)

This technique is similar to left-to-right subtraction except that it is done mentally and you only focus on the most significant one or two digits. For example, in estimating the difference between $359 and $274, you would first consider the hundreds place, then, realizing that the difference could either be 100 or 0 (depending on whether you need to regroup in the next place to the right), you inspect the next place and subtract there (Figure 10.33).

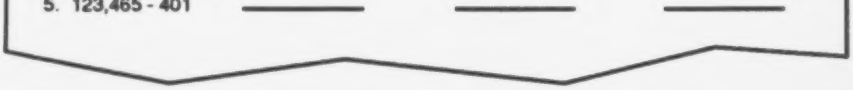
FIGURE 10.33
Front-end subtraction.


359 - 274
Since regrouping is needed, the difference in the hundred's place is zero. Subtract 7 from 15; the estimated answer is 80.
Difference is either 1 or 0 depending on next place

When teaching front-end subtraction to children, start with problems with numbers having the same number of places (e.g., 3481 – 1240). Present problems in vertical, horizontal, and verbal form. Ask them to calculate mentally the differences in the left place first, then the second to the left place. Have them write the answer from left to right, with zeroes in any remaining places.

After front-end subtraction involving numbers with the same number of places has been mastered, introduce problems with different numbers of places and those where regrouping will be needed to subtract in the second place to the right. See Figure 10.34 for an example worksheet.

FIGURE 10.34
Front-end subtraction worksheet.



Name ____________________

Directions: Write your estimate of the differences below. Find the answer with your calculator and decide how good your estimate was.

Problem	Estimate	Calculator	How good?
1. 3426 - 933	________	________	________
2. 500 - 37	________	________	________
3. 36,344 - 6300	________	________	________
4. 54,133 - 18,446	________	________	________
5. 123,465 - 401	________	________	________

■ Present problems in a variety of ways to encourage mental solution rather than the use of the standard pencil and paper algorithm. For example, present problems on an overhead and have the children write only the answers. Have one student read a problem to another and have that child determine the answer mentally and say it aloud.

■ Write several subtraction problems in missing-addend form and have students in the class estimate the missing addend.

35 + ____ = 57
27 + ____ = 93

Use calculators to check the reasonableness of the estimates and discuss strategies for narrowing in on the answer. Next, give students worksheets with similar problems and ask them to find the missing value in eight tries, writing each estimate and checking it with calculators.

14 + ____ = 98
14 + ____ = ______ (Try 1)
14 + ____ = ______ (Try 2)
14 + ____ = ______ (Try 3)
14 + ____ = ______ (Try 4) Etc.

■ Next, have students estimate differences involving problems in the standard, vertical format. Again, give them several chances to narrow in on the exact answer.

Check "12 + 68" with calculator. →

$$\begin{array}{r} 90 \\ -12 \\ \hline 68 \end{array} \qquad \begin{array}{r} 90 \\ -12 \\ \hline 78 \end{array} \qquad \begin{array}{r} 90 \\ -12 \\ \hline \end{array} \qquad \begin{array}{r} 90 \\ -12 \\ \hline \end{array}$$

Students should write each estimate and sum they get and you should check their work to see if they are developing successful strategies for narrowing in on the correct answer.

Verbal Problems (Objective M)

When dealing with verbal problems involving subtraction, be sure to include all four kinds of subtraction problems. Include problems with and without extraneous information. Use strategies presented in Chapter 5 to help your students become better problem solvers.

SAMPLE DRILL AND PRACTICE ACTIVITIES

Activity 10.1: Bundled-Stick Game

Objective: To provide practice with addition and regrouping at the concrete level.

Materials:

1. Bundled sticks
2. Dice

Methods:

1. Each player (up to 4) rolls the die in turn. The person who rolls the highest number begins.
2. The first player rolls the die and takes the corresponding number of single sticks (e.g., if he rolls a two, he takes 2 sticks).
3. The next player to the first player's left does the same, and so on, around in a clockwise fashion.
4. Play continues in this manner with each player keeping his own pile of sticks in front of him.
5. Whenever a player has 10 units he wraps a rubber band around them to form a bundle of ten.
6. The first player to be able to form a bundle of 10 tens (100) is the winner.

Variation 1: Choose another number like 250 as the goal.
Variation 2: Use the place-value chart instead of sticks.
Variation 3: Practice subtraction facts. Start with a bundle of 100 (10 tens). The first player to reach zero is the winner.

Activity 10.2: Addition Baseball

Objective: To practice addition facts.

Materials:

1. Flashcards with one-digit addition problems
2. A playing area with "bases"

Methods:

1. Divide the class into teams of 5 to 10. Teams decide on a batting order to which they must adhere.
2. Decide how many innings will be played.
3. The first batter comes up, and the teacher pitches by holding up a flashcard to the batter.
4. The batter has a given amount of time (perhaps 5 or 10 seconds) to give the right answer.
5. If the answer is correct, the batter goes to first base; if it is wrong, she is out and the next batter comes up.
6. The remainder of these rules of play are as in baseball (3 outs, etc.).
7. The team with the highest number of runs when the game is over is the winner.

Variation: Use subtraction flashcards.

Activity 10.3: Taking a Trip

Objective: To practice adding one-, two-, and three-digit numbers.

Materials:

1. A road map
2. Pencil and paper

Method: After an appropriate introduction to maps and reading distances on maps, give the student a road map and ask him to find the distance from one town to another taking several different routes. The techniques you have students use to add the distances will depend on your objective: they could use pencil and paper, mental arithmetic, calculators, the PRINT statement in Logo or BASIC, or a spreadsheet.

Activity 10.4: Place-Value Chart Game

Objective: To provide practice in multidigit subtraction on the place-value chart.

Materials:

1. Two dice in different colors (a tens' die and a ones' die)
2. Four-place (unlabeled) place-value chart

Methods:

1. Each player has his or her own place-value chart with a chip in the thousands' place.
2. Each player rolls a die to see who starts.
3. The first player rolls the two dice and removes the indicated number from her place-value chart (regrouping first if necessary)
4. The second player follows suit, and so on.
5. The first player to completely empty her place-value chart (when the number on the die is equal to, or larger than, the number indicated on the chart) is the winner.

Variation: Start with an empty place-value chart. Add the numbers indicated on the dice. The first player with a chip in the thousands' place is the winner.

Activity 10.5: Subtraction Race Cars

Objective: To provide practice with subtraction facts.

Materials:

1. Race Cars playing board (Fig. 20.6).
2. Two spinners with digits 0 – 9

Method:

1. Each player spins the spinner. The player with the highest number starts.
2. The player spins both spinners and subtracts the smaller number from the larger number. The player moves the number of spaces indicated by the difference. If the difference is 0, the player loses his turn.
3. Play proceeds in a counter-clockwise direction.
4. The first player to reach "Finish" is the winner.

SUGGESTED READINGS

Baroody, A. J. Children's difficulties in subtraction: Some causes and cures. *Arithmetic Teacher* 32(3), November 1984: 14–19.

Beattie, I. D. Modeling operations and algorithms. *Arithmetic Teacher* 33(6), February 1986: 23–28.

Campbell, P. F. What do children see in mathematics textbook pictures? *Arithmetic Teacher* 28(5), January 1981: 12–16.

Ewbank, W. A., and Ginther, J. L. Subtraction drill with a difference. *Arithmetic Teacher* 31(5), January 1984: 49–51.

Greene, G. Math-facts memory made easy. *Arithmetic Teacher* 33(4), December 1985: 21–25.

Grossman, A. S. A subtraction algorithm for the upper grades. *Arithmetic Teacher* 32(5), January 1985: 44–45.

Hutchinson, J. W., and Hutchinson, C. E. Homemade device for quick recall of basic facts. *Arithmetic Teacher* 25(4), January 1978: 54–55.

King, J. A. Missing addends: A case of reading comprehension. *Arithmetic Teacher* 30(1), September 1982: 44–45.

Litwiller, B. H., and Duncan, D. R. The extended subtraction table: A search for number patterns. *Arithmetic Teacher* 33(9), May 1986: 28–31.

Rathmell, E. C. Using strategies to teach the basic facts. *Developing computational skills* (1978 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1978.

Rightsel, P. S., and Thornton, C. A. 72 addition facts can be mastered by mid-grade 1. *Arithmetic Teacher* 33(3), November 1985: 8–10.

Thompson, C. S., and Dunlop, W. P. Basic facts: Do your children understand or do they memorize? *Arithmetic Teacher* 25(3), December 1977: 14–16.

Thompson, C. S. & Hendrickson, A. D. Verbal addition and subtraction problems: Some difficulties and some solutions, *Arithmetic Teacher* 33(7), March 1986: 21–25.

QUESTIONS

1. Describe a game designed to promote immediate recall of addition facts. One aspect of the game should be a time limit so that players cannot count or use another method to calculate the answers. See Chapter 20 for a discussion of basic game designs.

2. Write a computer program that will accept two addends, one place at a time, then will calculate and print the sum.

3. Describe an addition activity using a department store catalog, pencil and paper, and a calculator. Use only the dollar amounts in the catalog.

4. Discuss the merits of colored chips for the development of understanding of multidigit addition as compared to bundled sticks and the place-value chart.

5. Find a set of word problems in a subtraction chapter of a recent third or fourth grade mathematics textbook. Which of the *meanings* for subtraction are represented in the problems? How many problems for each meaning are there?

6. In this chapter, we discuss the use of the number line for teaching the take-away and missing addends meaning of subtraction. Can you think of a way to use the number line for teaching the set-comparison meaning?

7. Discuss an activity using manipulatives to help children discover that subtraction is not commutative.

8. Discuss how you could use calculators together with manipulative materials to develop understanding of multidigit subtraction.

9. Describe a game similar to Activity 8.1 for subtraction.

10. Read Campbell's article (see "Suggested Readings" above) on textbook illustrations, then find illustrations depicting mathematical operations in a first-grade mathematics textbook. Do you think that children would understand these illustrations better than the ones Campbell discusses? Why or why not? What could be done to help children understand textbook illustrations?

11. Solve the following problems using the equal additions method.

$$\text{a. } \begin{array}{r} 73 \\ \underline{-57} \end{array} \qquad \text{b. } \begin{array}{r} 164 \\ \underline{-\ \ 39} \end{array} \qquad \text{c. } \begin{array}{r} 209 \\ \underline{-136} \end{array} \qquad \text{d. } \begin{array}{r} 3000 \\ \underline{-\ \ 387} \end{array}$$

What advantages and disadvantages do you see with this method of subtraction? Should we replace our common algorithm with this technique?

12. Describe how you could use bundled sticks to represent the subset meaning of subtraction for the following problem:

$80 - 17 =$

13. Using the problem in question 12, how could you use bundled sticks to represent the equal addition meaning of subtraction?

CHAPTER

11

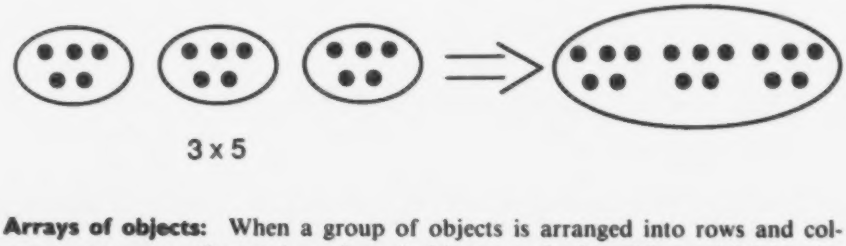
Multiplication & Division of Whole Numbers

MULTIPLICATION OF WHOLE NUMBERS

In one sense, multiplication is an extension of addition: any computational problem involving whole numbers can be solved through repeated addition. However, certain ideas involving multiplication are more complex than addition. There are three different types of real-life situations that suggest multiplication: (1) repeated joining of sets having the same number of objects, (2) arrays, and (3) cross-products.

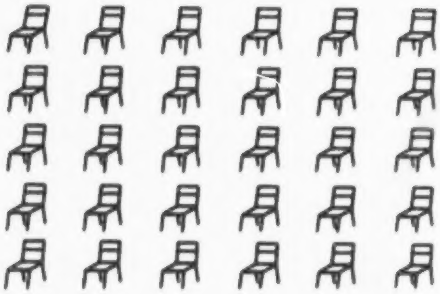
Repeated joining of sets: When we have a set of groups each containing the same number of items, we can find the total number of items by joining all the groups and counting or by repeatedly adding the number of items in each group (as many times as there are groups). This process is multiplication. An example is finding how many objects are in three sets, each containing five objects. See Figure 11.1.

FIGURE 11.1
The repeated joining of sets.



Arrays of objects: When a group of objects is arranged into rows and columns—an *array*—the total number of objects can be found by multiplying the number of rows by the number of columns. An example of an array, as shown in Figure 11.2, is when chairs are arranged in 5 rows with 6 chairs in each row.

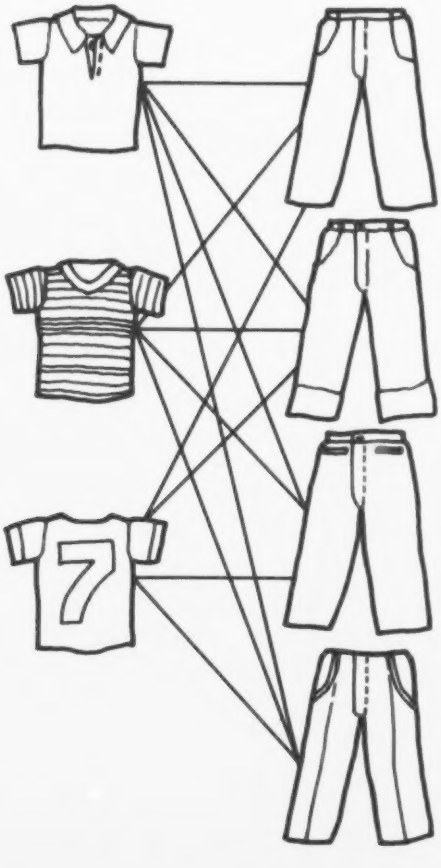
FIGURE 11.2
Arrays.



5 x 6

Cartesian products: If you have three different shirts and four different pairs of slacks, how many possible ways can you get dressed assuming the color combinations make no difference to you? This type of problem illustrates another meaning of multiplication—one that does not suggest addition. Looking at the possible ways two sets of objects can be combined is the *Cartesian product* meaning for multiplication. The drawing in Figure 11.3 illustrates how pictures might be used to find answers to Cartesian product problems.

FIGURE 11.3
Cross-products.



Behavioral Objectives for Multiplication of Whole Numbers

A. Make models for the set and array meanings of multiplication and use the models to find one-digit by one-digit products when verbally given.

B. When verbally given a real-life situation suggesting the set or array meanings for multiplication, solve the problem using either a physical model, a picture (drawing), or a meaningful computational technique (e.g., repeated addition).

C. Represent written one-digit by one-digit multiplication problems with a drawing and use the drawing to find the product.

D. Use a model to find answers to written one-digit products in either number-sentence or vertical form.

E. Use repeated addition to find answers to one-digit products.

F. State each basic multiplication fact (factors < 10) within 3 seconds when shown a flashcard.

G. Use a model to represent and find the product of a two- or three-digit factor and a one-digit factor.

H. Solve Cartesian product problems (given verbally) using physical models, pictures, or symbolic techniques.

I. Multiply a multidigit factor by a multidigit factor, listing all partial products.

J. Multiply a multidigit factor by any multidigit factor using the standard algorithm.

K. Solve verbal and applied problems involving multiplication.

L. Use a calculator to multiply two numbers.

M. Use the PRINT statement in Logo or BASIC to multiply two or more numbers.

N. Estimate the product of two numbers.

TEACHING MULTIPLICATION FACTS

The use of models for teaching meaning for multiplication may be even more important than their use for teaching children about addition. In addition, we can easily relate written problems to students' own experiences, but in multiplication that is not as easy. How many times do young children experience sets where all contain the same number of objects? How many times do they experience arrays where they must determine the total number of objects? These experiences are more likely to occur in the adult, working world than in the child's world.

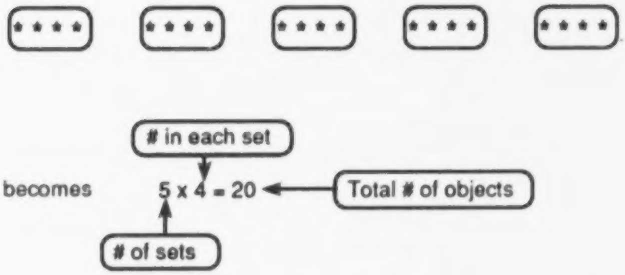
Single-Digit Multiplication (Objectives A–E)

The logical place to begin is with multiple joining of equivalent sets.

■ Using materials, explain to students the meaning of multiplication. Use loops of yarn and objects like poker chips or counting cubes to make several sets, all having the same number of objects. Ask the student how she could find out how many are in all the sets. Try to lead the student to seeing that repeated addition could be used, as could simply counting. Explain to the student that physical situations such as this represent multiplication—for example, when you have four groups of three, this becomes 4 × 3.

- Do similar activities using other materials. For example, popsicle sticks could be placed in open boxes to represent problems like 5 × 4 and 7 × 6.
- Relate the above activities to written multiplication problems (Figure 11.4).

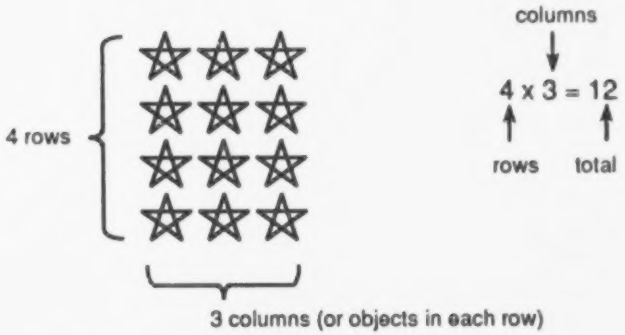
FIGURE 11.4
The "repeated addition of sets" meaning of multiplication.



# in each set
becomes
5 x 4 = 20
Total # of objects
# of sets

- Arrange objects such as chairs, checkers, and so forth, in arrays; point out the meaning of *rows* and *columns*. Point out the relationship between the array representation and the written multiplication problems, as in Figure 11.5.

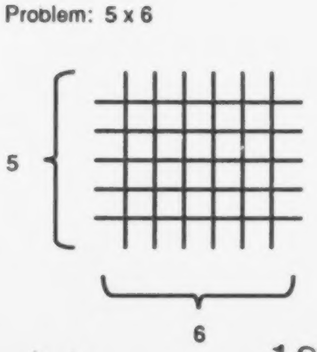
FIGURE 11.5
The "array" meaning of multiplication.



columns
4 rows
4 x 3 = 12
rows
total
3 columns (or objects in each row)

- Show the student a one-digit by one-digit written multiplication problem in either number-sentence or vertical form. Have the student draw arrays of dots or intersecting lines to solve the problem, as in Figure 11.6.

FIGURE 11.6
Intersecting lines illustrating multiplication.



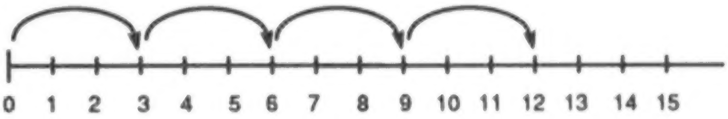
Problem: 5 x 6
5
6

■ Show the student how to use the number line to solve multiplication problems. Tell her that in the problem 4 × 3 = ?, the 4 indicates the number of jumps and the 3 indicates how long each jump is. Therefore, 4 × 3 on the number line means four jumps of three. (See Figure 11.7.)

FIGURE 11.7
Number line multiplication.

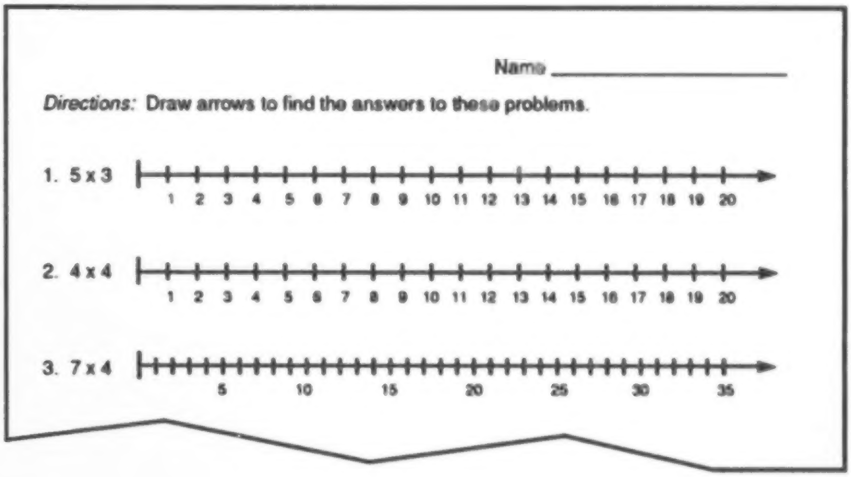
Problem: 4 x 3

Solution:



■ Give the student problems to solve and ask him to draw arrows on the accompanying number lines to find the answer, as in the sample exercise in Figure 11.8.

FIGURE 11.8
A worksheet for number line multiplication.



Name ______________

Directions: Draw arrows to find the answers to these problems.

1. 5 x 3
2. 4 x 4
3. 7 x 4

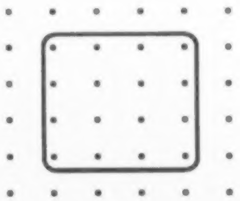
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
5 10 15 20 25 30 35

■ Show the student how to use a geoboard to find the answer to multiplication problems. (Place a rubber band around an array of nails with the appropriate number of rows and columns. Count the nails to determine the product. (See Figure 11.9.) Using the geoboard, investigate the relationship between pairs of products where the factors have been commuted (3 × 5 and 5 × 3).

FIGURE 11.9
Geoboard multiplication.

Problem: 4 x 4 = ☐

Solution:



■ Have students use Turtle Graphics to draw models of multiplication facts. Either ask them to draw sets of intersecting lines or arrays of dots to model products.

Repeated Addition (Objective E)

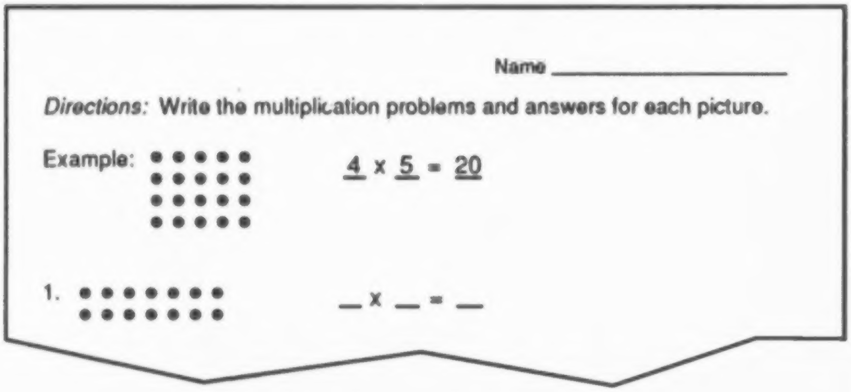
Demonstrate the use of repeated addition for finding products. For example, the answer to the problem 4 × 7 can be found by adding 4 sevens. Ask the students to solve multiplication problems on paper in this fashion.

■ In Chapter 3, the use of the constant function of the calculator to teach multiplication was discussed. Have the students enter the + 8 constant into the calculator, then add 5 eights to find the answer to 5 × 8. Have them check their answers by entering 5 × 8 = into the calculator. Let them calculate a variety of single-digit products using repeated addition on the calculator.

Memorizing Multiplication Facts (Objective F)

■ Show the student pictures of repeated sets and arrays of objects, as in Figure 11.10. Have him write the factors and the products.

FIGURE 11.10
Array multiplication worksheet.



Name ____________________

Directions: Write the multiplication problems and answers for each picture.

Example: 4 × 5 = 20

1. __ × __ = __

■ If the student has not already discovered the relationship between addition and multiplication, attempt to lead her to discover this relationship. One way to do this is to set up a manipulative problem suggesting multiplication, then ask her if she knows two ways to solve the problem. For example, make 5 sets each containing 6 objects. Two ways of expressing this problem are

6 + 6 + 6 + 6 + 6 = __ or 5 × 6 = __

■ Give timed drills where students must calculate a given number of products in a short period of time. Work on improving their rapidity and accuracy over a few weeks. Begin the drills with goals that are within reach of the students in order to give them a feeling of success.

■ Have the student complete tables like those in Figure 11.11. Point out the patterns of numbers in the table if the student does not see these for himself.

FIGURE 11.11
Multiplication tables.

X	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

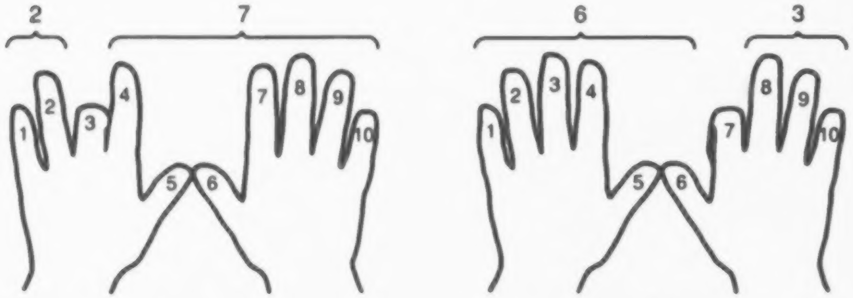
X	2	8	6	7	0	3	4	5	1
7									
6									
3									
0									
1									
2									
4									
5									
8									

■ With the help of your pupils make a large multiplication table. Ask them to look for patterns and help them make generalizations. For example, they might discover that

a. If zero is a factor, the product is zero.
b. If one is a factor, the product is the other factor.
c. If five is a factor, the product has 0 or 5 in the ones' place.
d. If eight is a factor, the product has 8,6,4,2,0 in the ones' place.
e. If nine is a factor, the sum of the digits in the product is divisible by 9.*

■ Show children how to use their two hands to find the products of the basic facts for nine. The fingers are numbered from 1 to 10. The finger representing the first factor is bent down. The number of fingers to the left of that finger represent the tens; the number of fingers to the right represent the ones.* Figure 11.12 illustrates this.

FIGURE 11.12
Finger multiplication.



3 x 9
2 tens and 7 ones (27)

7 x 9
6 tens and 3 ones (63)

---

*Activity by Jewell Garner.

PROPERTIES OF MULTIPLICATION

(1) Commutative Property of Multiplication

The commutative property of multiplication states that the product of two numbers is the same regardless of the order in which they are multiplied. Thus,

$$5 \times 4 = 4 \times 5 = 20 \quad \text{and}$$
$$298 \times 36 = 36 \times 298 = 10{,}728$$

Note that this property also applies to multiplication of fractions, decimals, integers, etc.

$$\tfrac{1}{2} \times \tfrac{3}{4} = \tfrac{3}{4} \times \tfrac{1}{2} = \tfrac{3}{8}$$
$$.25 \times 12 = 12 \times .25 = 3$$
$$-5 \times +3 = +3 \times -5 = -15$$

Children will not automatically and quickly discover that multiplication is commutative. In fact, many children will diligently compute or memorize pairs of "facts" like "3 × 6" and "6 × 3" separately, not realizing that if you know one of the pair, you know the other as well.

■ Children may be led to discover the commutative property by arranging multiplication problems so that the pattern is observed. For example, you might ask children to do a worksheet using manipulatives or calculators with adjacent pairs of single-digit multiplication facts, where the second pair has the same factors as the first, but in reverse order. Some sample problems:

5 × 4 = ___	4 × 5 = ___	1 × 8 = ___	8 × 1 = ___
3 × 7 = ___	7 × 3 = ___	6 × 5 = ___	5 × 6 = ___
6 × 5 = ___	5 × 6 = ___	2 × 7 = ___	7 × 2 = ___
		4 × 9 = ___	9 × 4 = ___

What pattern did you see?

(2) Identity Property of Multiplication

Another property that children can easily discover is that when you multiply a value by one, the result is the original value (e.g., 7 × 1 = 7). You may wish to present this concept to your class as a group or set up individual activities similar to the one discussed above involving the commutative property.

■ Prepare a worksheet of multiplication problems involving a factor of one and ask children to represent these problems with manipulative materials (e.g., 7 × 1 is one row with seven objects; 6 × 1 is one row with six objects). After students have determined answers to several problems, ask them what is the same in all the problems; next, ask them to state a generalization about them. If they have difficulty stating the generalization, ask, "What happens when we multiply any number by one?"

(3) Zero Property of Multiplication

Multiplication by zero is another property that may be discovered by manipulating concrete objects. Start by representing a product like 2 × 5 as two sets of five. Next represent and discuss 1 × 5. Finally, ask children to represent 0 × 5 and discuss. Repeat with the sequence 3 × 7, 2 × 7, 1 × 7, and 0 ×7. Next, have the students multiply various numbers by 0 using their calculators or using the PRINT statement in BASIC or Logo.

(4) Associative Property of Multiplication

The associative property means that when you are multiplying three numbers that are grouped in two different ways (since multiplication is a *binary* operation, you first multiply two of the factors, then multiply that answer by the third), the product is the same. Consider the following example: on the left side of the equal sign, the 6 and 3 are multiplied first since they are in parentheses, and the result is multiplied by 5 (6 × 3 = 18; then 18 × 5 = 90). On the right side of the equation, 3 and 5 are multiplied first for the same reason and the result (15) is multiplied by 6, also for an answer of 90.

$$(6 \times 3) \times 5 = 6 \times (3 \times 5)$$

As with previous mathematical properties, children may be led to discover the associative property. Explain that parentheses in an arithmetic sentence tell you which operation to do first—they change the usual left-to-right sequence of multiplying.

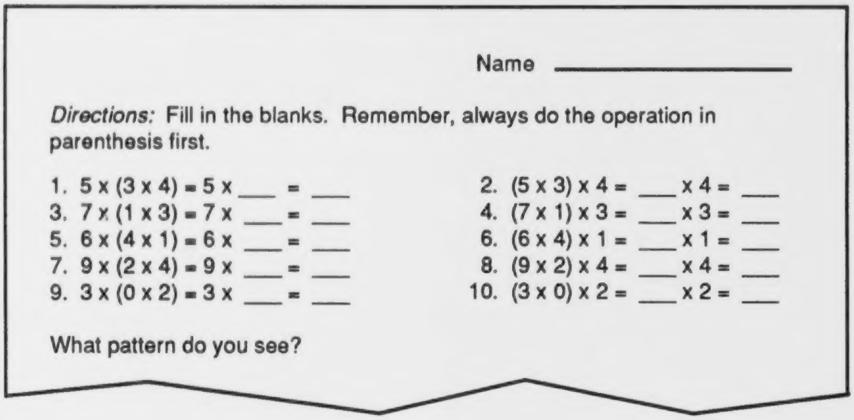
$$\underbrace{\underbrace{2 \times 3}_{6} \times 4}_{24}$$

(*Usual order*)

$$\underbrace{2 \times \underbrace{(3 \times 4)}_{12}}_{24}$$

(*With parentheses*)

Next have children solve sequences like those in Figure 11.13 using manipulatives, calculators, or mental techniques.



Name ______________

Directions: Fill in the blanks. Remember, always do the operation in parenthesis first.

1. 5 x (3 x 4) = 5 x ___ = ___
2. (5 x 3) x 4 = ___ x 4 = ___
3. 7 x (1 x 3) = 7 x ___ = ___
4. (7 x 1) x 3 = ___ x 3 = ___
5. 6 x (4 x 1) = 6 x ___ = ___
6. (6 x 4) x 1 = ___ x 1 = ___
7. 9 x (2 x 4) = 9 x ___ = ___
8. (9 x 2) x 4 = ___ x 4 = ___
9. 3 x (0 x 2) = 3 x ___ = ___
10. (3 x 0) x 2 = ___ x 2 = ___

What pattern do you see?

FIGURE 11.13
A worksheet for discovering the associative property of multiplication.

(5) Distributive Property of Multiplication

The distributive property states that when you multiply a number by the sum of two numbers, the same result is obtained whether you do the addition first and then multiply the sum by the number, or multiply the number by each of the numbers in the sum and then add the two resulting products.

$$a \times (b + c) = (a \times b) + (a \times c)$$
$$2 \times (3 + 4) = (2 \times 3) + (2 \times 4) = 14$$

This principle is fundamental to our standard multiplication algorithm:

$$\begin{aligned} 5 \times 25 &= 5 \times (20 + 5) \\ &= (5 \times 20) + (5 \times 5) \end{aligned}$$

The distributive property can also be useful in finding solutions when the multiplication facts are not known. For example, if you did not know 9×6 but knew that $9 \times 3 = 27$, you could use the distributive property to find the unknown product:

$$\begin{aligned} 9 \times 6 &= ? \\ 9 \times 6 &= 9 \times (3 + 3) \\ &= (9 \times 3) + (9 \times 3) \\ &= 27 + 27 = 54 \end{aligned}$$

Teaching the distributive property and its use in the multiplication algorithm is discussed below.

TEACHING MULTIDIGIT MULTIPLICATION

Models for Multidigit Multiplication (Objective G)

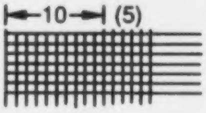
Oftentimes when teaching computational processes, we use materials to develop understanding of single-digit operations but neglect to do this for multidigit operations. We tend to assume that students will transfer earlier meaningful, manipulative experiences to operations with larger numbers. Usually this does not happen because it is months or years between the time single-digit operations and multidigit operations are introduced and used. If we teach multidigit computation only through drill and step-by-step processes, chances are that students will not understand the overall process or be able to apply their skills in problem-solving situations.

An excellent model for smaller two-digit products like 21 × 13 or 15 × 11 consists of centimeter grid paper and base-ten blocks. The model works without the blocks (i.e., you could draw the rectangles on the grid paper), but the blocks make the regrouping process much clearer. The model is set up as shown in Figure 11.14.

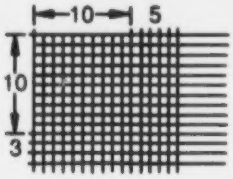
PROBLEM: 15 × 13 = __ (Figure 11.14)

FIGURE 11.14
Base-ten block multiplication.

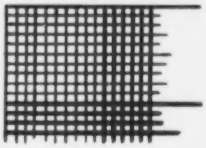
Step 1: Represent the tens and ones of 15 in the horizontal direction on the grid.



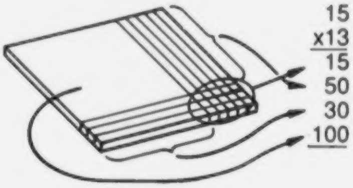
Step 2: Represent the tens and ones of 13 in the vertical direction on the grid.



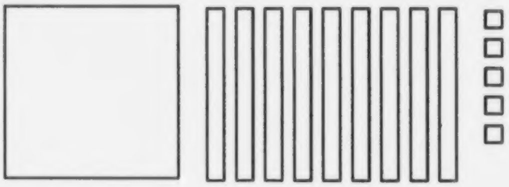
Step 3: Draw lines parallel to the sides for each ten and one shown in Steps 1 and 2.



The large squares correspond to the "flat", the rectangles to the "long", and the small squares to "units". At this point, the numeration blocks may be placed on the appropriate regions of the grid. Notice that the different sections of the representation correspond to the partial products of 15 x 13.



Step 4: Regroup if necessary and record the answer.

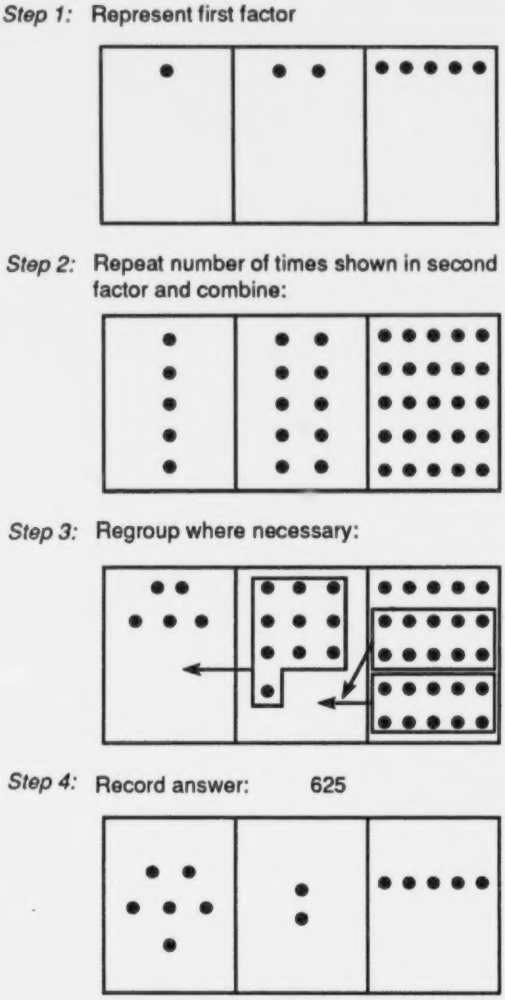


The place-value chart is useful for developing meaning for multidigit multiplication because the steps used in manipulating the materials parallel the steps in the multidigit multiplication algorithm. The major drawback of the place-value chart is that it is only practical when multiplying a multidigit factor by a single-digit factor (e.g., 125 × 5). Beyond that point it becomes very cumbersome.

The steps for multiplying a multidigit factor by a single-digit factor are as follows:

PROBLEM: 125 × 5 = __ (Figure 11.15)

FIGURE 11.15
Place-value chart multiplication.


Step 1: Represent first factor
Step 2: Repeat number of times shown in second factor and combine:
Step 3: Regroup where necessary:
Step 4: Record answer: 625

■ Demonstrate to the student(s) how to multiply two-digit numbers by one-digit numbers using the place-value chart. Ask them to do several examples. Repeat for three-digit by one-digit multiplication problems.

■ Relate the place-value chart activities to other manipulative experiences. For example, 125 × 5 could be solved using popsicle sticks (represent 125 five times, combine, and simplify).

■ Have the student carry out a variety of real-life, concrete activities involving multiplication. Allow them to use their own methods (e.g., repeated addition or place-value charts) for solving these problems, but attempt to help them understand that these are multiplication situations.

EXAMPLE 1: Find out how many pinto beans are in a jar using a small cup, by counting the number of beans in the cup and the number of cups that are in the large container, and then multiplying.

EXAMPLE 2: If each room in a school has 32 desks and there are 9 rooms in the school, how many desks would be in the school?
Give these problems to the students verbally and allow them to draw pictures, make models, or use repeated addition to solve them.

■ An important step in understanding multiplication is to relate manipulative experiences to computational techniques. Go through the steps of manipulating objects on the place-value chart and, at each step, perform the associated computation. The computational technique illustrated in Figure 11.16 is the method of recording all partial products.

PROBLEM: 6 × 209 = __ (See Figure 11.16 on the following page.)

Symbolic Activities (Objectives I–J)

As in learning addition and subtraction, students are helped by manipulations of physical objects, pictures, and other activities to understand the overall processes involved in multiplication. These activities help children understand pencil-and-paper computations as well as the mental algorithms we teach them.

Using Repeated Addition

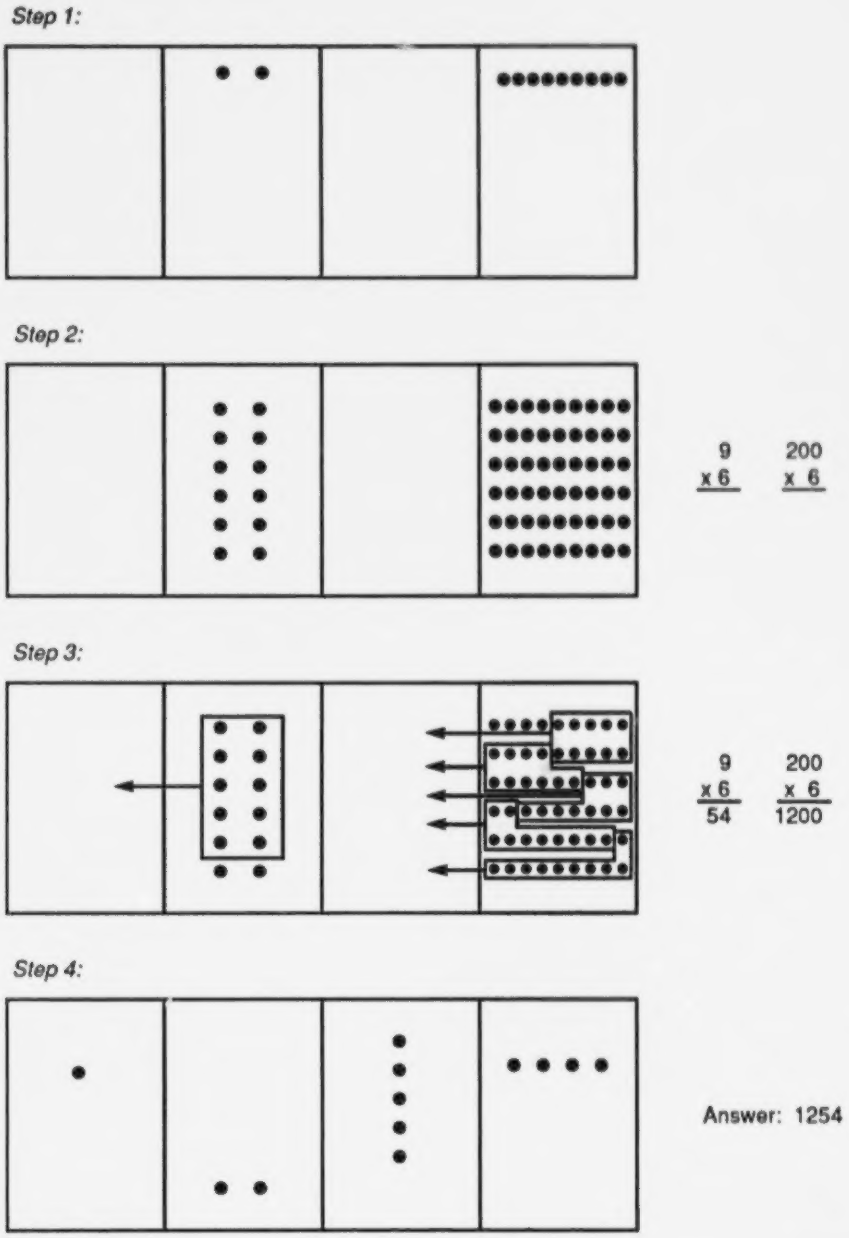
Repeated addition is usually used with basic multiplication facts to help students make the transition from addition to multiplication. By the time we get to multidigit multiplication, however, many children have forgotten that they did such activities. It is a good idea to spend a small amount of time doing repeated addition with pencil and paper, calculator, or computer to help reinforce the meaning of multiplication.

■ The best type of situation in which to use repeated addition is a computational problem involving a large and a small factor. An example is 3 × 125. The student's process would be

$$\begin{array}{r} 125 \\ +125 \\ \hline 250 \\ +125 \\ \hline 375 \end{array} \qquad \text{(125 is added 3 times)}$$

Repeated addition can be extended to problems involving larger factors as well. It is not always necessary to have the student actually carry out the repeated addition; instead, have her describe how the problem would be done.

FIGURE 11.16
Relating manipulations and partial products computations.


Step 1:
Step 2:
9 x 6
200 x 6
Step 3:
9 x 6 54
200 x 6 1200
Step 4:
Answer: 1254

- Have students find answers to problems such as 37 × 1568 using repeated addition with the constant function on the calculator. Have them compare answers. If they cannot agree upon the correct answer, have them use the multiplication key to determine the answer.

- To help reinforce the idea that multiplication problems may be solved through repeated addition and to experience use of looping and variables in Logo or BASIC, ask your students to write programs or procedures that accept two factors as inputs, then use repeated addition to find the product of the factors.

Teaching the Distributive Property

The distributive property is the foundation of the multiplication algorithm—it is what makes the process work. It is necessary that students understand this all-important property, but not necessary that they correctly use the term *distributive property*. What students must understand and remember is that each place in one number is multiplied by each place in the second number, and that all the partial products are added up. In order to help bring about this understanding of the following sequence, relate each step to manipulations with models.

PROBLEM: $8 \times 24 = \_\_$

STEP 1: Write 24 in expanded notation:

$$8 \times (20 + 4) = \_\_$$

STEP 2: Distribute the product across the sum of 20 and 4:

$$(8 \times 20) + (8 \times 4) = \_\_$$

An analogy might help make this step meaningful: If we have 20 girls and 4 boys in each of 8 classrooms, how many pupils will there be?

$$8 \times (20 \text{ girls} + 4 \text{ boys})$$

To solve this problem, you multiply the number of girls *and* boys by eight.

$$(8 \times 20 \text{ girls}) + (8 \times 4 \text{ boys})$$

STEP 3: Multiply: $8 \times 20 = 160$ and $8 \times 4 = 32$

STEP 4: Add: $160 + 32 = 192$

This procedure can be expanded to larger numbers but it is not advisable to ask students to develop the sequence for themselves. Rather go through several problems with students following and answering questions about the process. Later, have each student fill in blanks in the sequence.

EXAMPLE: Fill in the blanks.

$$\begin{aligned}
&= 22 \times 35 \\
&= 22 \times (30 + \underline{\qquad}) \\
&= (22 \times 30) + (\underline{\qquad} \times 5) \\
&= (20 + 2) \times 30 + (20 + 2) \times \underline{\qquad} \\
&= (20 \times 30) + (2 \times \underline{\qquad}) + (\underline{\qquad} \times 5) + (2 \times 5) \\
&= \underline{\qquad} + 60 + 100 + 10 \\
&= \underline{\qquad}
\end{aligned}$$

The point of this sequence is that 22 is *distributed* across the sum of 30 and 5. In the above sequence the distributive property is applied a second and third times. Certain students might have difficulty following this extended example, but all students should be taught the use of the distributive property for two-digit by one-digit multiplication.

After sufficient examples and practice, the student should be ready to transfer what she has learned to the use of the distributive property in vertical form. The steps below show the process.

PROBLEM:

$$\begin{array}{r} 45 \\ \underline{\times 6} \end{array}$$

STEP 1: Multiply 5 by 6:

$$\begin{array}{rr} 40 + & 5 \\ \times & 6 \\ \hline & 30 \end{array}$$

STEP 2: Multiply 40 by 6:

$$\begin{array}{rr} 40 + & 5 \\ \times & 6 \\ \hline 240 + & 30 \end{array}$$

STEP 3: Add

$$\begin{array}{rr} 40 + & 5 \\ \times & 6 \\ \hline 240 + & 30 = 270 \end{array}$$

Listing Partial Products

The most frequent causes of children's difficulties with multidigit multiplication are first, the inability to mentally regroup and second, difficulties with the placement of the *partial products*. As a method for developing overall understanding of the process of multiplication and as a transition to the traditional algorithm, have students work out and write down all partial products in a series of problems rather than mentally regrouping. The following sequence illustrates this method.

PROBLEM:

$$\begin{array}{r} 135 \\ \underline{\times\ 14} \end{array}$$

STEP 1: Multiply 4 × 5, record answer:

STEP 2: Multiply 4 × 30, record answer (*Note:* It is important that students be aware that you are multiplying **30** by **4** and not 3 by 4).

STEP 3: 4 × 100

STEP 4: 10 × 5

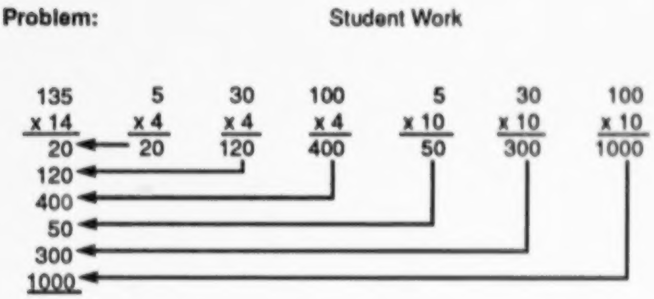
STEP 5: 10 × 30

STEP 6: 10 × 100

STEP 7: Add partial products.

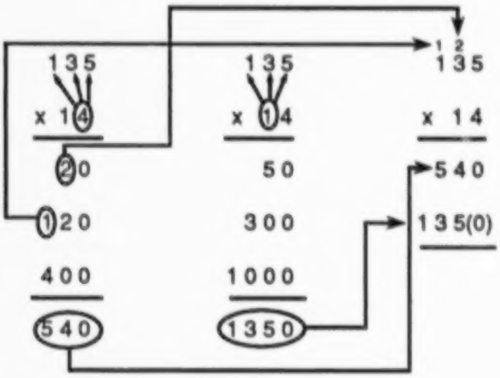
This process is illustrated in Figure 11.17. It is a good idea to have the student write *all* substeps.

FIGURE 11.17
Partial products multiplication.


Problem:
Student Work
135
x 14
20
120
400
50
300
1000
5
x 4
20
30
x 4
120
100
x 4
400
5
x 10
50
30
x 10
300
100
x 10
1000

After the student has mastered the *partial products* algorithm, show him the short cuts of the standard algorithm and how it relates to the partial products method, as shown in Figure 11.18. Obviously, it would be a good idea to start with less complex problems than the one in this example and progress to more difficult problems.

FIGURE 11.18
Short cuts.


1 3 5
x 1 4
2 0
1 2 0
4 0 0
5 4 0
1 3 5
x 1 4
5 0
3 0 0
1 0 0 0
1 3 5 0
1 2
1 3 5
x 1 4
5 4 0
1 3 5(0)

■ Introduce the symbol that is used for multiplying on the computer—the "*". Have students do a worksheet where the multiplication problems are written as Logo or BASIC PRINT statements (e.g., PRINT 22 * 14).

■ Have students use the calculator to find all the partial products in a series of multiplication problems. The student records these, then adds them up to find the product. The answer can be checked by entering the problem in the usual way. This technique forces him to concentrate on and understand the substeps in multiplication (e.g., in 256 × 38, the student must correctly enter 8 × 50 rather than 8 × 5 in order to arrive at the correct solution).

EXAMPLE:	256 × 38	2 5 6 × 3 8
STEPS:	1. 8 × 6 =	4 8
	2. 8 × 50 =	4 0 0
	3. 8 × 200 =	1 6 0 0
	4. 30 × 6 =	1 8 0
	5. 30 × 50 =	1 5 0 0
	6. 30 × 200 =	6 0 0 0
	7. Find sum	
	8. Check answer	

Verbal Problems (Objective K)

As with addition, the student may be helped in solving verbal problems involving multiplication if she is asked to make models of the problem. This may involve pictures of physical materials. Students should be encouraged to use whatever technique is most meaningful and accurate—repeated addition, listing partial products, or so on, in solving verbal problems.

Estimating Products (Objective O)

One technique for estimating products is to multiply the most significant (the leftmost) digits of the factors, then determine the place value of the product, and add an appropriate number of zeroes. To obtain a more accurate answer, you then adjust the product depending on the original values. Another strategy is to round the factors before multiplying, and then mentally to multiply the first digit of each factor, and determine the place value(s) of the result. These strategies are demonstrated in Figure 11.19.

FIGURE 11.19
Estimating products.

Multiplying Most Significant Digits

3468 x 490 = ?
1. Think, "3 x 4 = 12"
2. P.V. = 1000 x 100 = 100000
   Think "a hundred times a thousand is one hundred thousand," or "5 total zeroes"
3. Answer: 1200000 (1,200,000)
4. Adjust upward: 1,500,000 (?)

Rounding Factors

126 x 89 = ?
1. Round: 126 x 89 = 100 x 90
2. Multiply: 9 x 1 = 9
3. P.V. = 100 x 10 = 1000
4. Answer: 9000

DIVISION OF WHOLE NUMBERS

Division is not a difficult topic. Most children know something about division before they are introduced to it formally in school. For instance, they have distributed a bag of candy so that each recipient gets two pieces, or they have dealt a pack of cards so that each player has the same number.

The difficulty that we often associate with division is related to the long division algorithm. Indeed, this may be one of the most difficult procedures we ask children to master in elementary school. The National Council of Teachers of Mathematics and other organizations and individuals have questioned the value of spending so much time attempting to teach the long division algorithm when calculators and computers are available to divide for us. There are so many more important things to teach in the elementary curriculum, and there are other, easier algorithms for dividing a number by another using pencil and paper. Rather than focusing on long division, we will be looking at a variety of methods for developing understanding of division and for calculating answers to division problems.

What is division? Division is the operation that allows us to find the missing factor when we know one factor and the product in a multiplication situation.

$4 \times \_\_ = 24$

Thus, division is the inverse operation of multiplication, just as subtraction is the inverse of addition.

Two concrete situations give rise to division. The first is *measurement division*, or repeated subtraction. In this situation, you begin with a certain quantity and ask how many sets of a given quantity you can remove from the larger set. For example, suppose you won $1000 in the Lottery and decided to give it away in increments of $50 to your friends until the money was gone. How many happy friends would you have? The second concrete situation is *partitive division*. In this meaning of division, you have a set of a given size and ask how many objects should be in each set if you want to divide the larger set into a given number of equal-sized sets. For example, John has a bag of 30 candies and wants to distribute the candy to five friends so that each has the same number of candies. How many candies will each get?

Behavioral Objectives for Division of Whole Numbers

A. Partition a given number of objects equally among a given number of sets and determine how many objects are in each set (no remainder).

B. Measure a given number of objects into sets of equal and given size and determine how many sets of objects there are (no remainder).

C. Measure or partition objects with remainders.

D. Represent a written division fact (divisor and quotient 10 or less, no remainder) with either a partitioning or measurement model.

E. Be able to state or write within a given time period basic division facts (divisor and quotient 10 or less, no remainder) that are shown in written form.

F. Use a model to find answers to written division problems (dividend < 200, divisor < 100).

G. Use repeated subtraction to solve division problems (with or without remainder).

H. Divide a multidigit number by a multidigit number using subtractive or standard algorithm.
I. Solve verbal and applied problems involving division.
J. Use a calculator to find the quotient of two numbers.
K. Use the PRINT statement in Logo or BASIC to find the quotient of two numbers.
L. Estimate the quotient of two numbers.
M. Solve problems involving three or more operations (+, −, ÷, ×) on whole numbers.
N. Use correct order of operations to find answers to exercises on whole numbers involving multiple operations and parentheses.

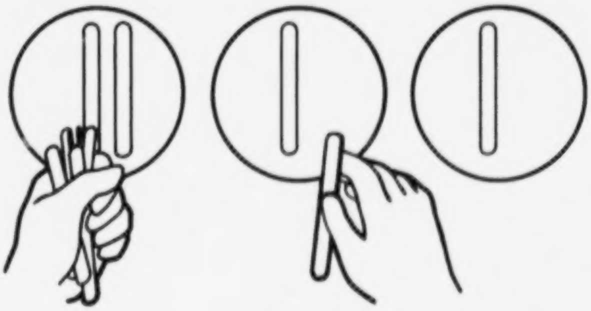
TEACHING DIVISION FACTS

Activities With Single Objects (Objectives A–D)

Activities with objects should involve both the *measurement* and *partitioning* meanings of division. They should also include dividends not exactly divisible by the divisors (i.e., problems with remainders).

■ Demonstrate to pupils how to partition a group of objects (between 6 and 20) equally into containers or loops of yarn—place one in each container, place a second in each container, and so on, until all the counters are partitioned. Be sure to have the children check to see that each container has the same number of objects or counters, and then count the number of objects in each set to discover the quotient. (Figure 11.20.)

FIGURE 11.20
Partitioning.



Next have the children themselves do partitioning activities with counters and containers. For example, have each child take 5 paper cups and count out 15 beans. Next ask him to partition the beans into the cups. Have him check to see if the beans were partitioned correctly. Finally, have him state or write the quotient. Repeat with other quotients with no remainder.

■ Demonstrate to the class how to measure a set of objects. Draw 10 "bags" on an overhead transparency. Tell the class that you have 24 quarters and you want to give 4 quarters to children who come to your door at Halloween until you run out.

How many children will get quarters? Place "quarters" (use dried beans, transparent overhead "money," or real quarters) 4 at a time in the bags and ask the class to state the answer. Repeat for other quotients. Next, have children use containers or loops of yarn, and count objects to find answers to measurement problems with no remainders. Say, for example, "Here are 48 chips. I want you to find out how many sets of 6 you can make with them."

At this point, you may wish to discuss the relationship between division and subtraction. When you have 35 candies and want determine how many people will get candy if you give 5 to each, you are repeatedly subtracting 5 from the original set. Asking how many people will get candy if you have 35 candies and give 5 to each is the same as asking how many times you can subtract 5 from 35.

The next step is to associate partitioning and measurement activities with written division problems. After an appropriate discussion, present problems with single-digit quotients and no remainders for the students to solve using counters and containers.

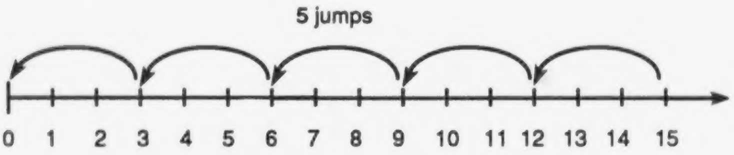
- Do activities similar to those discussed above but where the number of objects is not divisible by the number of containers. Ask the student to say how many objects are in each set and how many are left over. For example, show him the number sentence 15 ÷ 4 = __, take fifteen single objects and say, "This is the number of beans we start with and this (4) is the number of sets we are dividing them into." You should go through the activity of dividing 15 by 4 with the materials, state the number of elements in each set, identify how many are left over, and then write the answer. Include both measurement and partitioning examples.

- Present problems to the student in number-sentence form and have her use materials to solve the problem. Supply the student with more than enough single objects and containers and allow her to decide how to use them to solve the problems.

- Present similar problems in long-division form (e.g., $5\overline{)20}$) and have the student use materials to solve the problems, then record the answer in the usual place. Do similar activities with remainders. Have the students write the answer and the remainder in both number-sentence and standard (long-division) form.

- Use the number line to illustrate the *measurement* meaning of division. To illustrate 15 divided by 3, for example, start at 15 and subtract units of 3 until you reach zero or a number less than 3, as in Figure 11.21. The number of "jumps" is the quotient. The number line shows the inverse relationship between division and multiplication, and also the relationship between division and subtraction.

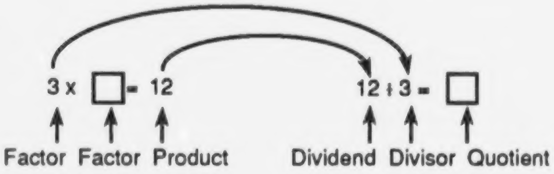
FIGURE 11.21
Number line division.



Learning Division Facts (Objective E)

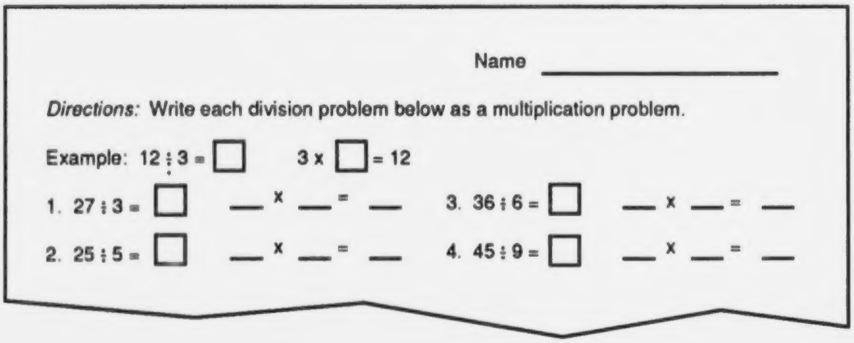
After the basic meanings of division have been developed using materials, we can point out the relationship between division and multiplication, as in Figure 11.22. Make a model for a multiplication problem and discuss the fact that if you know the number of sets and the number in each set but not the total number, you are multiplying. If you know the total number and either the number of sets or the number in each set is missing, you are dividing.

FIGURE 11.22
The relationship between division and multiplication.



■ In order to reinforce the relationship between division and multiplication, ask students to rewrite a number of division number sentences as multiplication number sentences with a missing factor. A sample worksheet is shown in Figure 11.23.

FIGURE 11.23
Division–multiplication worksheet.



Name ______________

Directions: Write each division problem below as a multiplication problem.

Example: 12 ÷ 3 = □ 3 x □ = 12

1. 27 ÷ 3 = □ ___ x ___ = ___ 3. 36 ÷ 6 = □ ___ x ___ = ___

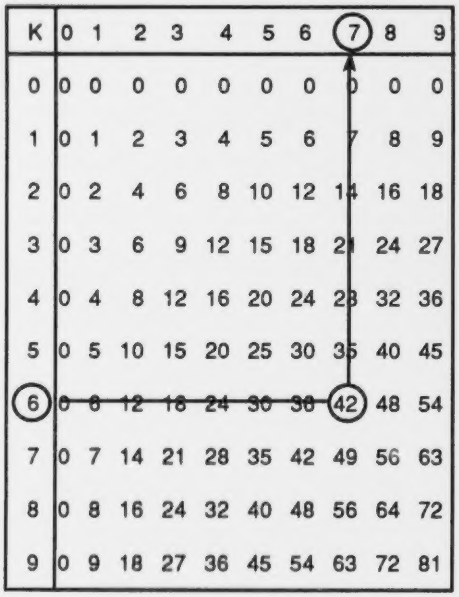
2. 25 ÷ 5 = □ ___ x ___ = ___ 4. 45 ÷ 9 = □ ___ x ___ = ___

■ Point out the relationship between division in standard (long-division) form with multiplication in standard (vertical) form:

$6\overline{)30}$ $\begin{array}{r} 6 \\ \times\ \underline{\quad} \\ 30 \end{array}$

The multiplication table is quite useful for solving division problems and it helps reinforce the relationship between the two operations (see Figure 11.24). Students find one factor (the divisor) in the left-hand column and the dividend in the body of the table. The quotient is at the top of the column containing the dividend.

FIGURE 11.24
Multiplication table for division.



K	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Problem: 42 ÷ 6

Find 6 in the left column and 42 in that row.

The quotient is at the top of the column.

- Work on speeding students' responses to basic division facts by using flashcards. If the student has difficulty, again point out the relationship between multiplication and division. For example, say (when you show the problem $6\overline{)30}$), "Six times what number is 30?"
- Give students a worksheet with 20 basic division facts and see how many they can do correctly in $2^1/_2$ minutes. Give them timed tests periodically, decreasing the amount of time allowed and increasing the percentage correct.

PROPERTIES OF DIVISION

Division is not *commutative*

$$24 \div 6 \neq 6 \div 24$$

nor is it *associative*

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2).$$

The *unit* for division is 1 ($15 \div 1 = 15$); however, the unit property is only valid if you are dividing by one, not if one is being divided by another number ($1 \div 15 \neq$

15). The *distributive* property is valid for division only if the division is to the right of the distribution:

$$(8 + 6) \div 2 = (8 \div 2) + (6 \div 2).$$

The *zero property* is valid when dividing zero by another number ($0 \div 256 = 0$). Dividing by zero is illegal since there is no value that will make this number sentence true:

$$0 \times \_\_\_\_\_ = 64$$

MULTIDIGIT DIVISION

Manipulative Activities for Multidigit Division (Objective F)

Several activities with manipulative materials can be used to help children understand multidigit division. However, since most of these are relatively complex and involve several steps, they will not be of value to every student; many students will benefit more from symbolic development of meaning, using repeated subtraction or other techniques.

If you decide that it is important to teach children the long-division algorithm, you may want to consider the following manipulations with beans on a place-value chart (see Figure 11.25) to develop meaning. As a parallel activity, you may wish to change 6 dollar bills, 2 dimes, and 5 pennies so that the money can be divided equally among 5 friends (using only dollars, dimes, and pennies). A similar sequence can be followed with bundled sticks, base-ten blocks, or colored chips.

PROBLEM: $5\overline{)625}$ (Figure 11.25)

Symbolic Activities (Objectives G–H)

(1) Repeated Subtraction

Some textbooks have students use repeated subtraction to solve simple division problems in order to develop understanding. They explain that $12 \div 3 = \_\_\_\_$ (or $3\overline{)12}$) can be interpreted to mean, "How many times can you subtract 3 from 12?" In symbolic form,

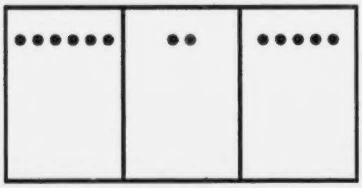
$$3\overline{)12} \quad \text{or} \quad \begin{array}{r}12\\ \underline{-3}\\ 9\end{array} \quad \begin{array}{r}9\\ \underline{-3}\\ 6\end{array} \quad \begin{array}{r}6\\ \underline{-3}\\ 3\end{array} \quad \begin{array}{r}3\\ \underline{-3}\\ 0\end{array} \quad \text{or} \quad \begin{array}{l}\text{You can subtract 3 from 12}\\ \textit{4 times.}\end{array}$$

They then ask children to solve a few problems similarly. Since, by the time children get to multidigit division they will have forgotten these activities, we need to remind them of the relationship between division and subtraction by asking them to solve a few multidigit division problems using subtraction.

Be sure to pick the examples carefully to avoid frustration. The best problems for this sort of activity, if using pencil and paper, are those [illegible]ng 3 through 8

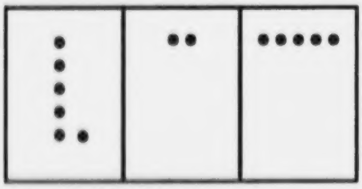
FIGURE 11.25
Place-value chart division.

Step 1: Represent the problem on the place-value chart:

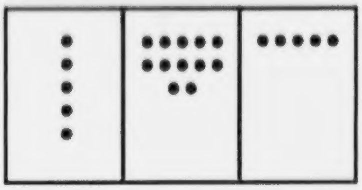


Process: Attempt to divide the beans in each place into five equal sets.

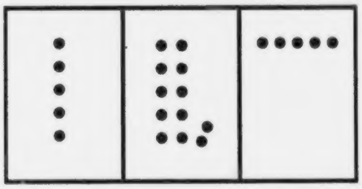
Step 2: Make 5 rows in the hundred's place with one bean in each row:



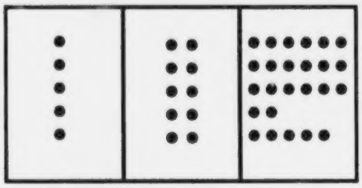
Step 3: Exchange the left-over hundred for 10 tens:



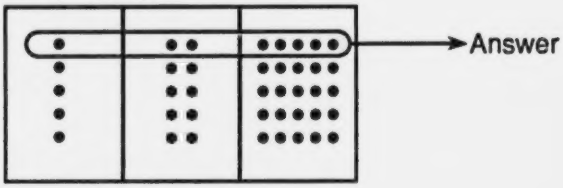
Step 4: Arrange the tens into five equal rows:



Step 5: Regroup:



Step 6: Arrange the ones in five equal groups:



Step 7: Write answer: 1 2 5

subtractions. Example: 72 ÷ 24 = ____. Also include problems having *remainders*. (The remainder is what is left over when the last difference is larger than 0 and smaller than the divisor.)

$$48 \div 15 = \_\_ \text{ or } \underbrace{\begin{array}{r}48\\-15\\\hline 33\end{array} \quad \begin{array}{r}33\\-15\\\hline 18\end{array} \quad \begin{array}{r}18\\-15\\\hline 3\end{array}}_{\text{3 times}}$$

Last answer (3) is smaller than 15; therefore it is the remainder.

- The calculator allows you to expand to larger and more complex numbers in these activities.

PROBLEM: 1225 ÷ 25 = ?

SOLUTION: 1225 − 25 = = = = = = = = . . . until the display reads zero. The student counts the number of times "=" is hit and that is the answer (quotient).

Have the students work division problems using repeated subtraction on the calculator, have them record their answers on a worksheet, and then have them check their answers by dividing in the usual fashion. (Note: the discussion in Chapter 3 gives hints on making sure students actually go through the repeated subtraction process rather than using the ÷ key on their calculator.)

- Introduce the use of the "/" sign for doing division on the computer. Discuss the meaning of problems like 12/6 (12 ÷ 6). Give students a worksheet with problems stated as PRINT statements and ask them to calculate and write the answers (e.g., PRINT 144/12). Then have them use the immediate mode in either Logo or BASIC to check their answers.

- Have students write a Logo or BASIC program that accepts a dividend and divisor, uses repeated subtraction to find the quotient, then prints the quotient and remainder, if any.

Dividing with a Calculator (Objective J)

It is important for us to spend class time teaching children to divide numbers using the calculator. As preparation, they should have solved division problems in the algebraic (number-sentence) form

256 ÷ 16 = ____

because they will enter the division problems in that order on the calculator

256 ÷ 16 =

An important consideration when using the calculator to divide is the remainder. Any time you use the calculator to divide numbers that are not exactly divisible, your answer will contain a decimal. Ideally (if they are following a curriculum that prepares them for today's technological society), children will have been exposed to decimal numeration and understand the meaning of numbers like 25.5 before they work with multidigit division. Realistically, however, few textbooks or mathematical skills lists follow this sequence. Thus, you need to decide how to handle this dilemma. A "non-solution" would be to limit the operands to those with whole-number quotients since we should be using the calculator to expand children's mathematical horizons and using realistic numbers in their problem-solving activities.

The best solution is to give students some introductory instruction on the meaning of decimals through manipulatives, grids, and number lines. Techniques for accomplishing this are discussed in Chapter 12. After students have some understanding of decimals, they can use the calculator to convert decimal remainders to whole number remainders by multiplying the decimal portion of their answer by the divisor. For example:

PROBLEM: 25 ÷ 8 =
SOLUTION: 25 ÷ 8 = 3.125 (*display*)

To convert .125 to a remainder, multiply

.125 by 8 (*the divisor*)
.125 × 8 = 1 (*displays the remainder*)

The problem that remains is how to handle repeating decimals. When dividing 88 by 3 on the calculator, you get 29.3333333. Then, if you multiply .3333333 by 3, you get .9999999 since most calculators truncate repeating decimals (though some more expensive calculators round or store several places internally beyond those displayed). Children should be given instruction and practice in dealing with repeating decimal quotients on the calculator. Of course, we assume that they have had previous rounding experience.

(2) The Subtractive Algorithm

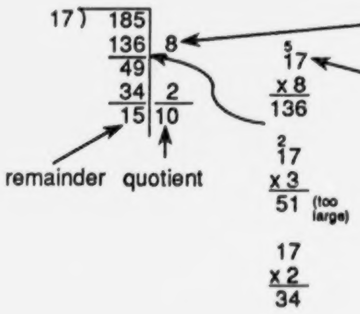
The subtractive algorithm has been taught for some time in many school districts around the country instead of the "standard," long division algorithm. Recently some textbooks have revived this division technique because of its advantages over the standard algorithm. These advantages are

- It is more meaningful because the user is constantly aware of the overall process of division (in the example in Figure 11.26, the learner estimates the quotient $17\overline{)185}$, rather than dividing 18 by 17, then the product of that estimate and 17 is subtracted from 185).
- The user does not have to be good at estimating quotients in order to use this technique, since even a poor estimate of the quotient moves one toward a solution.
- There are not as many steps to remember.
- It can be used with varying degrees of efficiency by novices through experts.

It is the author's opinion that all children should be taught this algorithm in lieu of the long division algorithm. The steps are shown in Figure 11.26.

PROBLEM: $17\overline{)185}$ (Figure 11.26)

FIGURE 11.26
Subtractive algorithm.



1. Estimate the number of times you can subtract 17 from 185. Write the estimate here.
2. Multiply this estimate by the divisor. Write the product so that it may be subtracted from 185 (136 is 8 seventeens).
3. Subtract 136 from 185.
4. Estimate the number of times you can subtract 17 from the remaining amount. Note that if your estimate is too large, try again.
5. Subtract this product from what is left.
6. Repeat Steps 4 and 5 until the remaining value is zero or less than the divisor.
7. Add the numbers to the right of the horizontal bar (this is the number of times 17 has been subtracted from 185).
8. You may wish to ask students to write the quotient in the usual place above the 185.

Essentially, there are three steps that the student repeats until the remainder is zero or less than the divisor [(1)estimate the quotient, (2) multiply, and (3) subtract)]. He then adds the number of times the divisor was subtracted to obtain the quotient. If the student is good at estimating, this algorithm is just as efficient as the usual method of division.

When teaching children to use this technique, start by giving them simple division problems in the subtractive form and have them use repeated subtraction to find the quotient. Have them write a 1 to the right of the vertical line each time they subtract the divisor. To find the quotient, have them add the ones:

$$\begin{array}{rl}
12\overline{)48} & \\
\underline{12} & 1 \\
36 & \\
\underline{12} & 1 \\
24 & \\
\underline{12} & 1 \\
12 & \\
\underline{12} & \underline{1} \\
0 & 4
\end{array}$$

Next, have them estimate and then subtract multiples of the divisor:

$$\begin{array}{rr}
3\overline{)96} & \\
\underline{18} & 6 \\
78 & \\
\underline{60} & 20 \\
18 & \\
\underline{18} & \underline{6} \\
0 & 32
\end{array}$$

In the beginning, their estimates will be poor and their strategies inefficient. Be sure to evaluate their work and lead them to become more efficient users of the algorithm. Of course, early experiences should be with two-digit by one-digit division (no remainders).

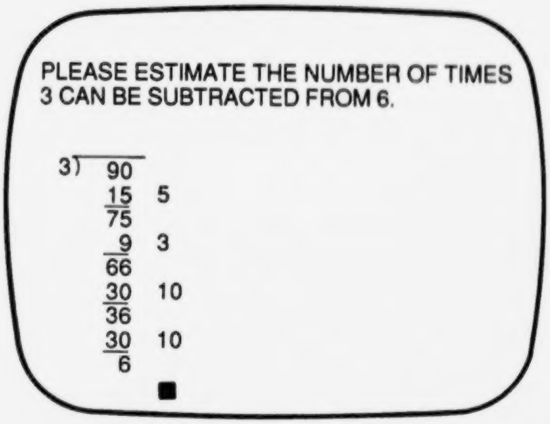
Next, teach them to become more efficient users of the algorithm by multiplying the divisor by 10 and multiples of 10 (later, with more complex problems, multiply by 100, 1000, etc.), and then subtracting these amounts from the dividend:

```
15)1800           15)1800
   1500  100         1500  100
    300               300
    150   10          300   20
    150                 0  120
    150   10
      0  120
```

Gradually introduce and give them practice with larger numbers. Include problems with remainders. Originally, express the answers as remainders, later as fractions or decimals.

The program called SUBTRACT in Appendix E of this book is designed to give practice and feedback in the use of the subtractive algorithm. It places a division problem on the screen, like the one in Figure 11.27, and asks the student to estimate how many times the divisor can be subtracted from the dividend. If the student's estimate is too large, she is told to try again, otherwise the student's estimate and the product of the estimate and the divisor are written in the appropriate place, etc. When the amount remaining is zero, the student is asked to enter the quotient.

FIGURE 11.27
A screen display of the SUBTRACTIVE program.



```
PLEASE ESTIMATE THE NUMBER OF TIMES
3 CAN BE SUBTRACTED FROM 6.

3) 90
   15   5
   75
    9   3
   66
   30  10
   36
   30  10
    6
        ■
```

(3) Long Division

If you decide to teach your students to use the long-division algorithm, you will need to make the transition from the subtractive to the "standard" technique. Start by concentrating on the place value of the first digit in the quotient. Figure 11.28 demonstrates that this can be found by multiplying the divisor by successive powers of 10. In this example, the quotient is less than 1000 but greater than 100.

FIGURE 11.28
Place value of first digit of quotient.

$$18\overline{)3942}$$

$18 \times 10 = 180$

$18 \times \underbrace{100} = \boxed{1800}$ → Smaller than dividend

$18 \times 1000 = \boxed{18{,}000}$ → Larger than dividend

→ The place value of the quotient is 100.

Next, find the multiple of 100 that belongs in that place (Figure 11.29). Of course, a student's estimation and mental arithmetic skills are handy here. It will help him to round the dividend to 3900 (or better yet, round the divisor to 20 and dividend to 4000) before estimating the quotient. You may also wish to allow your students to use their calculators to check the products of the divisor times the estimated quotients after they have estimated the partial quotients. (See Figure 11.30.)

FIGURE 11.29
Exact value of first digit of quotient.

$$\begin{array}{r} 200 \\ 18\overline{)3942} \end{array}$$

$18 \times 100 = 1800$

$18 \times \underbrace{200} = \boxed{3600}$ → Smaller than dividend

$18 \times 300 = \boxed{5400}$ → Larger than dividend

→ Write this value over dividend

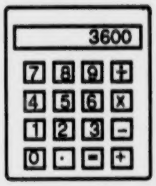
FIGURE 11.30
Using a calculator for substeps of long division.

$$\begin{array}{r} 200 \\ 18\overline{)3942} \\ \underline{3600} \end{array}$$

Sequence of keystrokes to find first digit in quotient:

18 $\boxed{\times}$ 10 $\boxed{=}$ (180)
18 $\boxed{\times}$ 100 $\boxed{=}$ (1800)
18 $\boxed{\times}$ 1000 $\boxed{=}$ (18,000)
18 $\boxed{\times}$ $\underbrace{200}$ $\boxed{=}$ $\underbrace{(3600)}$

Write over dividend
Write under dividend



When the exact value in the left-most place in the quotient has been found, the student writes the complete product of divisor times that value under the dividend and subtracts. Next, have students use estimation to find the largest multiple of 10 that, multiplied by 18, gives a number smaller than 342, and write this value above the previous partial quotient. (See Figure 11.31.)

FIGURE 11.31
Finding the second most significant digit.

$$\begin{array}{r} 10 \\ \underline{200} \\ 18\overline{)3942} \\ \underline{3600} \\ 342 \\ \underline{180} \\ 162 \end{array}$$

$18 \times \underbrace{10} = \boxed{180}$ → Smaller than 342

$18 \times 20 = 360$

→ Write over the 200

Repeat for the remainder of the places. Given their experience with expanded notation, the students should be able to convert the values to a single quotient, as shown in Figure 11.32. Of course, you will want to have them use their calculators to multiply their quotient by the divisor to check the accuracy of their computation.

Initial presentation of this technique for dividing should be done at the chalkboard or on an overhead with numerous examples. Next, children should have ample opportunity to practice, starting with problems having few places ($9\overline{)189}$) and values where it is relatively obvious how to round when estimating quotients ($24\overline{)528}$).

Those who become proficient with this technique may discover (or be led to discover) that they need only write the first digit of the partial quotients and that, when they have written all partial quotients, they have their answer. They may also be able to get by without writing the zeros below the problem. Do not progress too quickly to this final form of the algorithm, however, nor insist that all students master it. As was mentioned earlier, this is one of the most complex tasks we ask elementary children to learn. It is better to be taught a division technique that they can master, remember, and use than one that is very efficient but that they find difficult to use to obtain correct answers.

FIGURE 11.32
Determining the quotient.

$$\begin{array}{r} 9 \\ 10 \\ \underline{200} \\ 18\overline{)3942} \\ \underline{3600} \\ 342 \\ \underline{180} \\ 162 \\ \underline{162} \\ 0 \end{array}$$

Quotient = 200 + 10 + 9 = 219

Remainders

Many of the concrete examples of division that children deal with involve a remainder. For example, if I have 29 candies to divide equally among 6 friends, 5 will be left over for me to keep. In early division experiences with manipulatives and symbols, we simply express answers with remainders or leftovers. (e.g., "Each person gets 5 beans and there are 3 left over"). The way remainders in more advanced problems are treated, however, depends on the real-life situation out of which the problem arises. If you are asked how many $9 record albums you can buy for $50, you ignore the remainder. On the other hand, if four cooks divided 14 pounds of sugar equally, each would get $3^{1}/_{2}$ pounds. When developing meaning for and applying division, we need to be sure to give many real-life examples including division situations with all three types of remainders, illustrated in Figure 11.33.

FIGURE 11.33
Remainders.

Remainder Expressed	Remainder Ignored	Expressed as Fraction
$\begin{array}{r} 4 \\ 6\overline{)29} \\ \underline{24} \\ 5 \end{array}$	$\begin{array}{r} 5 \\ 9\overline{)50} \\ \underline{45} \\ 5 \end{array}$	$\begin{array}{r} 3 \\ 4\overline{)14} \\ \underline{12} \\ 2 \end{array}$
Answer: 4R5	Answer: 5 records	Answer: $3\frac{1}{2}$ pounds

Estimating Quotients (Objective L)

Children's experiences with the subtractive algorithm will help them when it comes to estimating quotients since that is what they are doing when they use the algorithm. Additional experience in estimating quotients should be provided, however.

One way to convert division problems into ones that are easier to handle mentally is to delete zeroes. We know that if we multiply or divide both the numerator and denominator of a fraction by the same value, the fraction is equivalent to the original:

$$\frac{3000}{4000} \div \frac{1000}{1000} = \frac{3}{4}$$

If we think of our division problems as fractions (32,458 ÷ 4162 = $^{32,458}/_{4162}$), we can divide the numerator and denominator by a power of 10.

$$\frac{32,458 \div 1000}{4162 \div 1000} = \frac{32.458}{4.162}$$

Rounding the numerator and denominator to the nearest unit gives us a division fact (32 ÷ 4). The same technique (see Figure 11.34) can be done in the context of a division problem or number sentence.

FIGURE 11.34
Deleting places.

$$4\cancel{162}\overline{)32\cancel{458}} \Rightarrow 4\overline{)32}$$

Cross out the same number of digits in the dividend and divisor.

$$30\cancel{98} - 6\cancel{21} \Rightarrow 30 \div 6$$

- Give students problems in long-division form (e.g., $25\overline{)1236}$) and ask them to write answers without doing any computations. Next, have them use calculators to compute the exact answers. Finally, ask them to evaluate the accuracy of their answers.

- Give them problems in the *missing factor* form and ask them to estimate the missing factor, then multiply the two factors using the calculator. See if they can find the missing factor within a given number of tries. See Example 5 in Chapter 3.

- Give students division problems in a variety of forms, including verbal, and ask them to estimate the quotient. Have them use their calculators to check the accuracy of their answers.

- Another strategy for estimating quotients is to change the values in the problem to similar values which are divisible. For example, in estimating 353 ÷ 62, 353 could be changed to 360 and 62 to 60. The resulting problem, 360 ÷ 60 can be divided mentally and the resulting answer is very close to the precise answer.

- After introducing this technique, give children practice problems with *compatible* numbers (i.e., numbers that are adjacent to those which are mentally divisible because the first digit or two in the divisor and dividend are basic facts). Have them use calculators to find the precise quotient, then have them evaluate the accuracy of their answers.

Verbal Problems (Objective I)

Give students a variety of measurement and partitioning problems in verbal form. In presenting students with these verbal problems, mix in problems requiring other operations and/or requiring the student to select relevant information. If a student knows before he reads the problem that he will "divide the big number by the little number," not much problem solving takes place.

Order of Operations (Objectives M–N)

When a person uses a calculator or a computer to do more than one operation, she needs to know how the device will evaluate the expression. For example, if she enters 4 + 5 × 6 = into different calculators, some will display 54, others will display 34. If she includes the program line PRINT 4 + 5 × 6 in a program, the output will be 34 because the computer does the multiplication before the addition.

The standard way of carrying out a series of operations in a mathematical expression is first to do multiplications and divisions in left-to-right order, and then to do the additions and subtractions in left-to-right order. Fig. 11.35 shows the standard order for two expressions. Logo and BASIC both follow this order.

FIGURE 11.35
An example of order of operations.

$$3 + \underbrace{5 \times 7}$$
$$\underbrace{3 + 35}$$
$$38$$

$$\underbrace{2 \times 6}_{12} + 8 - \underbrace{3 \times 2}_{6} + \underbrace{12 \div 4}_{3} \times 2$$

New Expression: $12 + 8 - 6 + \underbrace{3 \times 2}_{6}$

New Expression: $\underbrace{12 + 8} - 6 + 6$

$$\underbrace{20 - 6} + 6$$
$$\underbrace{14 + 6}$$
$$20$$

Most inexpensive calculators evaluate mathematical expressions from left to right as they are entered; however, some of the more expensive calculators use the correct mathematical order. The difference is shown in Figure 11.36.

FIGURE 11.36
The evaluation of expressions by calculators.

Calculator 1 (Inexpensive Calculator)

$3 \boxed{+} 4 \boxed{\times} 5 \boxed{=}$ 35 (display)

Calculator 2 (Scientific Calculator)

$3 \boxed{+} 4 \boxed{\times} 5 \boxed{=}$ 23 (display)

If you wish to change the standard order of evaluation in an expression to be calculated use parentheses. Parentheses override the standard order of operations because the expression in parentheses is treated as a single number. In BASIC or Logo, therefore, for the expression PRINT (3 + 4) * 5, the operation inside the parentheses is simplified to 7, which is then multiplied by 5. If there are parentheses inside of parentheses, the innermost are evaluated first. Figure 11.37 gives examples of how expressions with parentheses are evaluated. Note that most cheap calculators do not have parentheses while scientific calculators do.

FIGURE 11.37
The evaluation of expressions with parentheses.

$$\begin{array}{l}
\underbrace{(3+4)}\times 5 \\
\underbrace{7 \quad \times 5} \\
35
\end{array}
\qquad\qquad
\begin{array}{l}
(\underbrace{(3+2)}\times 6+8)+5 \\
(\underbrace{5\times 6}+8)+5 \\
(\underbrace{30+8})+5 \\
38+5 \\
43
\end{array}$$

■ To introduce the idea that not all calculators evaluate expressions the same way, have children bring calculators from home. Make sure that at least one calculator is available that evaluates expressions in the correct mathematical order. Have children enter a variety of expressions where the order of evaluation is different (e.g., "5 + 2 × 6") for scientific calculators and inexpensive calculators. Discuss the results and introduce children to the standard order of operations. Give them worksheets with expressions containing more than two operations and have them determine the answers. You may have them use PRINT statements in Logo or BASIC or scientific calculators to determine the correct answers.

■ Discuss methods for obtaining the correct answers involving multiple operations on inexpensive, nonscientific calculators. For example, evaluate the expression arising from the following problem.

PROBLEM: John bought 2 candy bars costing 45 cents each and Sally bought two colas costing 65 cents each. How much did they spend?

SOLUTION: 2 × 45 + 2 × 65 (or 2 × .45 + 2 × .65)

One method would be to calculate each of the products, and write the first one down on paper, then add the first to the second. The second method would be to use the calculator's memory:

2 × 45 = M+ 2 × 65 = M+ MR
or 2 × 45 = M+ 2 × 65 = + MR =

Give students a variety of verbal problems involving multiple operations and ask them to use the calculator to solve them. If you sense that some students are having problems, have them write down the sequence of operations they enter into the calculator so that you can diagnose their errors.

SAMPLE DRILL AND PRACTICE ACTIVITIES

Activity 11.1 Foot Race

Objective: Use manipulatives for finding single-digit products given verbally.

Materials:

1. Geoboard and rubber bands
2. "Race Cars" game board (see Chapter 20)

3. Stack of problem cards with multiplication facts
4. Dice
5. Answer sheet

Methods:

1. Each player rolls a die to see who starts. The highest number starts.
2. Players take turns in a clockwise rotation drawing cards, showing the product with a rubber band in the geoboard, and counting the pegs to determine the answer. If the player either demonstrates the product incorrectly or gives the product (answer) incorrectly, he loses his turn and the next player plays. If the answer is correct, he rolls the die and moves his marker the appropriate number of spaces on the playing board. The first player to arrive at "Finish" is the winner. (Example, 3 × 5 would be shown as a rubber band around a 3 by 5 array, and the student would say "fifteen" as the total.)
   Variation: Use problem cards with division facts.

Activity 11.2 Multiplication Concentration

Objective: State single-digit multiplication facts.

Materials: Tagboard squares with products and answers, as in Figure 11.38.

FIGURE 11.38
Sample concentration squares.

4 x 5	20	9 x 4	36	7 x 7	49
3 x 8	24	7 x 6	42	9 x 6	54
7 x 5	35	8 x 5	40	7 x 4	28
6 x 5	30	8 x 8	64	6 x 9	54

Methods:

1. Mix up tiles and arrange them into a square or rectangular array with problems and answers turned face down.
2. Take turns turning over pairs of tiles. If they make a match (e.g., "4 × 5" and "20"), keep both tiles and play again. If they do not, return them to their original position and the next player plays. When all tiles have been picked up, the person with the most tiles is the winner.
3. If students are unsure of their multiplication facts, provide them with a calculator or manipulative materials to find the answers.

Activity 11.3 Multiplication Bingo

See Chapter 20, Figure 20.3–20.5.

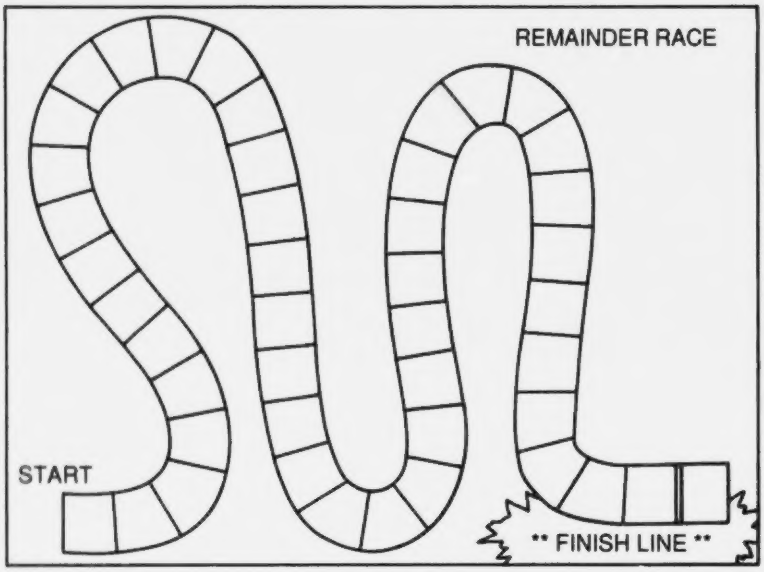
Activity 11.4: Remainder Race

Objective: Divide two-digit numbers by one-digit numbers with remainders.

Materials:

1. Remainder Race playing board (Figure 11.39)
2. Division cards
3. Die

FIGURE 11.39
A Remainder Race Board.



$2\overline{)\,5}$
$2\overline{)\,6}$
$2\overline{)\,7}$
$2\overline{)\,8}$
$2\overline{)\,9}$
$2\overline{)\,10}$
$2\overline{)\,11}$
$2\overline{)\,12}$
$2\overline{)\,13}$
$2\overline{)\,14}$

$2\overline{)\,15}$
$2\overline{)\,16}$
$2\overline{)\,17}$
$2\overline{)\,18}$
$2\overline{)\,19}$
$3\overline{)\,6}$
$3\overline{)\,7}$
$3\overline{)\,8}$
$3\overline{)\,9}$
$3\overline{)\,10}$

$3\overline{)\,11}$
$3\overline{)\,12}$
$3\overline{)\,13}$
$3\overline{)\,14}$
$3\overline{)\,15}$
$3\overline{)\,16}$
$3\overline{)\,17}$
$3\overline{)\,18}$
$3\overline{)\,19}$
$3\overline{)\,20}$

$4\overline{)\,9}$
$4\overline{)\,10}$
$4\overline{)\,11}$
$4\overline{)\,12}$
$4\overline{)\,13}$
$4\overline{)\,14}$
$4\overline{)\,15}$
$4\overline{)\,16}$
$4\overline{)\,17}$
$4\overline{)\,18}$

$4\overline{)\,19}$
$4\overline{)\,20}$
$4\overline{)\,21}$
$4\overline{)\,22}$
$4\overline{)\,23}$
$5\overline{)\,11}$
$5\overline{)\,12}$
$5\overline{)\,13}$
$5\overline{)\,14}$
$5\overline{)\,15}$

$5\overline{)\,16}$
$5\overline{)\,17}$
$5\overline{)\,18}$
$5\overline{)\,19}$
$5\overline{)\,20}$
$5\overline{)\,21}$
$5\overline{)\,22}$
$5\overline{)\,23}$
$5\overline{)\,24}$
$5\overline{)\,25}$

$6\overline{)\,13}$
$6\overline{)\,14}$
$6\overline{)\,15}$
$6\overline{)\,16}$
$6\overline{)\,17}$
$6\overline{)\,18}$
$6\overline{)\,19}$
$6\overline{)\,20}$
$6\overline{)\,21}$
$6\overline{)\,22}$

$6\overline{)\,23}$
$6\overline{)\,24}$
$6\overline{)\,25}$
$6\overline{)\,26}$
$6\overline{)\,27}$
$7\overline{)\,15}$
$7\overline{)\,16}$
$7\overline{)\,17}$
$7\overline{)\,18}$
$7\overline{)\,19}$

$7\overline{)\,20}$
$7\overline{)\,21}$
$7\overline{)\,22}$
$7\overline{)\,23}$
$7\overline{)\,24}$
$7\overline{)\,25}$
$7\overline{)\,26}$
$7\overline{)\,27}$
$7\overline{)\,28}$
$7\overline{)\,29}$

Methods:

1. Children roll a die to see who starts.
2. The first player draws a card, works the problem, then moves her marker the number of spaces indicated by the remainder.
3. The next player takes his turn, and so on.
4. Play continues until a player (the winner) reaches the "finish."

Activity 11.5: Geography Number Puzzle

Objective: Divide a three- or four-digit number by a one- or two-digit number, no regrouping.

Materials: See Figure 11.40.

FIGURE 11.40
A geography number puzzle.

Geography Number Puzzle Name: ____________________

$\overline{30}$ $\overline{20}$ $\overline{72}$ $\overline{23}$ $\overline{56}$ $\overline{25}$ $\overline{14}$ $\overline{46}$ $\overline{56}$ $\overline{20}$ $\overline{30}$ $\overline{72}$

$\overline{24}$ $\overline{30}$ $\overline{14}$ $\overline{51}$ $\overline{20}$ $\overline{45}$

A. $7\overline{)98}$	B. $6\overline{)144}$	C. $8\overline{)344}$	D. $12\overline{)276}$	E. $13\overline{)728}$
F. $15\overline{)390}$	G. $9\overline{)495}$	H. $18\overline{)846}$	I. $17\overline{)340}$	J. $10\overline{)250}$
K. $4\overline{)504}$	L. $15\overline{)675}$	M. $12\overline{)324}$	N. $14\overline{)644}$	O. $11\overline{)792}$
P. $13\overline{)702}$	Q. $9\overline{)315}$	R. $6\overline{)180}$	S. $10\overline{)640}$	T. $21\overline{)861}$
U. $5\overline{)590}$	V. $13\overline{)858}$	W. $11\overline{)429}$	Y. $16\overline{)528}$	Z. $9\overline{)459}$

Methods: Find the answers to the quotients in the puzzle. Look for the quotient above. If you find it, write the letter of the problem on the line above the answer. After all the letters are filled in, you will find the name of a Latin city.

SUGGESTED READINGS

Bates, T. and Rousseau, L. Will the real division algorithm please stand up? *Arithmetic Teacher* 33(7), March 1986: 42–45.

Beattie, I. D. Modeling operations and algorithms. *Arithmetic Teacher* 33(6), February 1986: 23–28.

Bolduc, E. J. Jr. The monsters in multiplication. *Arithmetic Teacher* 28(3), November 1980: 24–26.

Bruni, J. V., and Silverman, H. J. The multiplication facts: Once more, with understanding. *Arithmetic Teacher* 23(6), October 1976: 402–409.

Dunkels, A. More popsicle-stick multiplication. *Arithmetic Teacher* 29(7), March 1982: 20–21.

Falco, T. Multiplication clue: A game activity for the classroom. *Arithmetic Teacher* 31(2), October 1983: 36–37.

Hall, W. D. Division with base-ten blocks. *Arithmetic Teacher* 31(3), November 1983: 16–20.

Hazekamp, D. W. Teaching multiplication and division algorithms. *Developing Computational Skills* (1978 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1978.

Heddens, J. W., and Lazerick, B. So 3 'guzinta' 5 once: So what? *Arithmetic Teacher* 29(7), November 1977: 576–578.

Hendrickson, A. D. Verbal multiplication and division problems: Some difficulties and some solutions. *Arithmetic Teacher* 33(8), April 1986: 26–33.

Kalin, R. How students do their division facts. *Arithmetic Teacher* 31(3), November 1983: 16–20.

Katterns, B. and Carr, K. Talking with young children about multiplication, *Arithmetic Teacher* 33(8), April 1986: 18–21.

Kulm, G. Multiplication and division algorithms in German schools. *Arithmetic Teacher* 27(9), May 1980: 26–27.

Laing, R. A., and Meyer, R. A. Transitional division algorithms. *Arithmetic Teacher* 29(9), May 1982: 10–12.

Meyer, R. A., and Riley, J. E. Multiplication games. *Arithmetic Teacher* 33(8), April 1986: 22–25.

Quintero, A. H. Children's conceptual understanding of situations involving multiplication, *Arithmetic Teacher* 33(5), January 1986: 34–37.

Quintero, A. H. Conceptual understanding of multiplication: Problems involving combination. *Arithmetic Teacher* 33(3), November 1985: 36–39.

Stuart, M., and Bestgen, B. Productive pieces: Exploring multiplication on the overhead. *Arithmetic Teacher* 29(5), January 1982: 22–23.

Weiland, L. Matching instruction to children's thinking about division, *Arithmetic Teacher* 33(4), December 1985: 34–35.

Winters, M. J. Teaching mathematics with microcomputers: Middle grades. *Arithmetic Teacher* 30(6), February 1983: 28–29.

QUESTIONS

1. Discuss the limitations of models for teaching multidigit multiplication.
2. Given the abundance of calculating devices in our society, how proficient should our students become in long multiplication of large numbers?
3. Obtain a current third- or fourth-grade math textbook and explore how models are used to develop understanding of single-digit multiplication. Repeat for multidigit multiplication. What types of applications for multiplication are presented in the text?
4. Take a walk through your neighborhood, a shopping mall, or your college and look for situations or objects that suggest multiplication (e.g., seats in a classroom or windows in a tall building). Discuss how such a walk could be done with a class of students, how the examples of multiplication situations could be recorded and brought back to the classroom, and how you could use such an activity as a foundation for further work in multiplication.
5. Write a Logo or BASIC program that accepts two factors as inputs and then finds and prints the product of the two numbers using repeated addition in a REPEAT statement (Logo) or a FOR-TO, NEXT loop (BASIC).
6. Solve the following long-division problem, describing in writing *each* step as you complete it. Repeat using the subtractive algorithm, then discuss the relative difficulty of the two algorithms.

   $27\overline{)3045}$

7. How do textbooks teach division? Look at the models and algorithms taught in several third- and fifth-grade textbooks.
8. Make some suggestions for dealing with the remainder when using a calculator to do repeated subtraction for division.
9. Read Laing and Mayer's article on transitional division algorithms, then discuss the possibility of not teaching long division at all vs. teaching the subtractive algorithm as a transition to the long division algorithm.
10. Write a Logo or BASIC program that accepts a divisor and dividend, then uses repeated subtraction to find and print the quotient and remainder.

CHAPTER

12

Understanding Fractions & Decimals

One of the more important aspects of literacy is the ability to read and understand various kinds of numerals. A freeway sign might state that the next rest stop is in $2^1/_4$ miles, and in order to read your odometer to determine if you are getting close, you must realize that $2^1/_4$ is between 2.2 and 2.3. A recipe might ask you to add $^1/_2$ tsp. of salt or $2^1/_3$ cups of flour. If you want to use your calculator to add $2^7/_8$ inches to $5^1/_4$ inches, you must either know the decimal equivalents of these two values or know how to find them (preferably, using the calculator). The headlines state that 35.7% of the country's population supports the president's policies in Central America or that the unemployment rate is $7^1/_2$%. Without the ability to read and understand the meaning of fractional, decimal, and percentage numerals, a person would have difficulty coping in our society.

This chapter presents methods for developing understanding of the meaning of numerals like $^1/_4$, 2.37, and related concepts. Percentage numerals will be covered in Chapter 14. In order to help children to become functionally literate and to use calculators and computers within and outside of the classroom, it is advised that we introduce them to simple fractions (*not* operations on fractions, however) in the first grade and that we begin their instruction in decimal numeration after their mastery of three-digit whole number numeration and mixed fractions, probably toward the end of second grade for most children. Instruction in *operations* on fractions, decimals, and percentages should be delayed until later grades.

For the most part, the numbers (both whole and nonwhole) we deal with in the elementary grades are rational. What are *rational numbers*?

> A rational number is any number that can be expressed as $^a/_b$ (positive or negative) where a and b are whole numbers and b is not zero.

In other words, a rational number is any number that can be expressed as a fraction. That includes whole numbers (4 can be written as $^8/_2$, $^4/_1$, etc.) and decimals (.34 can be expressed as $^{34}/_{100}$). It also includes negative numbers like −5 and −33.33333. . . .

Fractions and decimals are different ways of representing the same number ideas. For example, the numeral used to represent the picture in Figure 12.1 could be any of the following numerals: $^1/_8$, $^5/_{40}$, .125, .1250000.

Several different concrete situations are represented by fractions and decimals.

1. The partitioning of a whole: A whole object is divided into congruent parts as in Figure 12.2. The denominator is the total number of subparts and the numerator is the subset being considered.
2. Part of a discrete set of objects: Part of a set of objects is identified as pictured in Figure 12.3. The number of objects in the entire set is the denominator and the subset being considered is the numerator.

Note that with a set of separate objects, it is not necessary that all the objects be the same size or shape. We may say that $^2/_9$ of the dogs in the kennel are black even though dogs come in different sizes and shapes.

FIGURE 12.1
Pictorial representation of rational numbers.

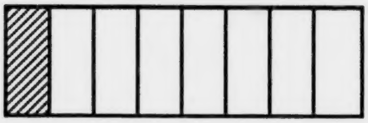


FIGURE 12.2
The partitioning of a whole object.

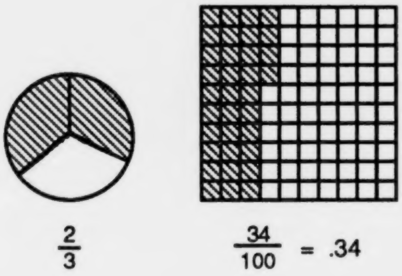
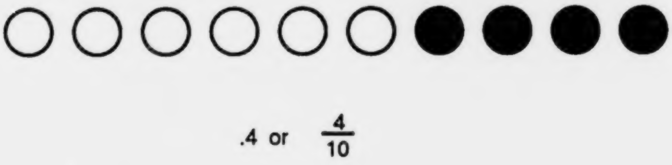


FIGURE 12.3
A subset model.



Students need to understand that each of the above situations suggests a fractional or a decimal numeral. Beginning experiences can and should include whole numbers in fractional settings and fractions larger than 1. The student should also have many experiences that lead to an understanding of equivalent fractions, decimals, and percentages.

Behavioral Objectives for Fraction and Decimal Meanings and Numerals

A. Say or write the correct fractional name (proper, improper, or mixed) when shown a concrete model and vice versa.
B. Shade rectangles, circles, hexagons, or squares to represent a fraction verbally stated by the teacher and state fraction represented by pictorial model.
C. Manipulate or identify models to demonstrate the equivalence of two fractions.
D. Identify the location of proper fractions, improper fractions, and mixed numerals on number lines.
E. Write a decimal numeral in tenths or hundredths when shown a model or vice versa.
F. Shade a rectangle divided into 100 congruent sections to represent a written decimal in thousandths.
G. Shade a rectangle divided into 10 or 100 congruent sections to represent a written decimal fraction in tenths or hundredths followed by a fraction (e.g., $.37\frac{1}{2}$) or write a numeral from the model.
H. Write three decimal fractions equal to a given decimal fraction.
I. Identify the place values of digits in a decimal numeral.
J. Write a decimal numeral in expanded notation.
K. Identify the smallest or largest of several decimal numerals.
L. Round a decimal to a given place.
M. Regroup a decimal fraction in a given place.
N. Convert a decimal to a fraction and vice versa.
O. Given a frequently-used fraction, state the decimal equivalent in a short period of time and vice versa.

TEACHING FRACTIONS

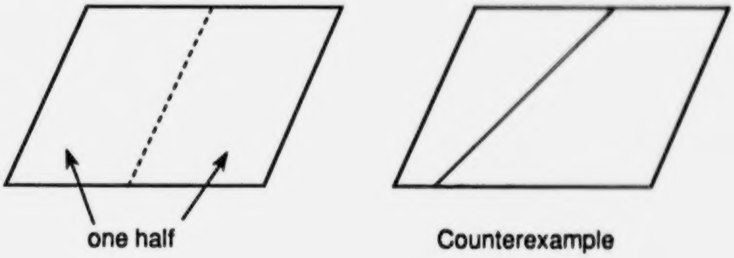
Models for Fraction Concepts (Objectives A–B)

A prerequisite to developing the part-whole meaning of fractions is the idea of *congruence*. Activities with cut-out parts can be used to develop and reinforce this concept. Another idea that needs to be developed is that of the *unit*. Seven parts cannot represent a fraction until they are compared with the "whole" or unit.

■ Demonstrate the meaning of *one half* by cutting rectangular pieces of cardboard into two congruent parts, then superimposing one piece over the other and examining the two together. Also, use similar materials to demonstrate counterexamples of one half, as in the example in Figure 12.4.

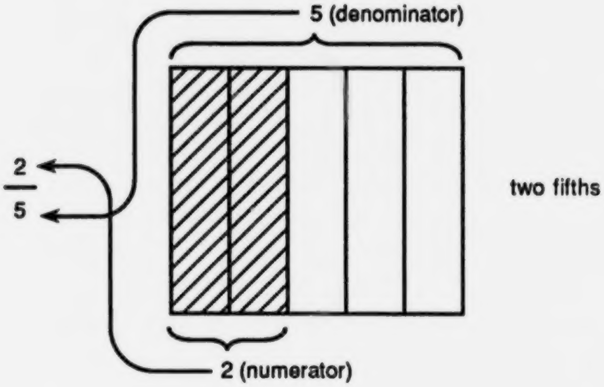
Demonstrate and discuss a variety of models and counterexamples of fractions with 1 as the denominator, such as $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, using a variety of shapes—rectangles, circles, hexagons, etc. Compare the cut-out pieces to a whole. For example, if you use construction paper cut into congruent pieces to represent $\frac{1}{6}$, place one of the pieces over a whole sheet of construction paper and discuss.

FIGURE 12.4
Models and counterexamples of one half.



- Discuss the names given to fractions. Point out that *half* means that the whole is divided into 2 congruent parts, *third* means that it is divided into 3 congruent parts, *fourth*, and so on. The word *one* in "one-half" tells you how many of these pieces you are using. Figure 12.5 shows the narrative used to explain the meaning of fractions and the verbal names for them.

FIGURE 12.5
Names for fractions.



Relate the models to written numerals. Ask the students to write numerals for fractions represented with the models and vice versa.

- Represent fractions with numerators other than 1—$^{2}/_{3}$, $^{3}/_{5}$, etc. Have students state and write the fractions that the models represent.

- Have students use one-inch grid paper to represent a variety of fractions given in written form, as illustrated in Figure 12.6. First, have them draw around the number of squares representing the denominator; next have them color the squares representing the numerator; and finally have them cut out their representation of the fraction.

- Use hexagons to demonstrate dividing a shape into three, six, and twelve congruent sections. It may be helpful to draw dotted and solid lines between vertices to guide where the student should subdivide the shapes to represent fractions, as in Figure 12.7.

- Use the manipulatives and shapes discussed above to represent wholes (e.g., if *fifths* are cut from $8^{1}/_{2}$ by 11 inch construction paper, use an uncut sheet to represent the *whole*). Have students discuss, write, and represent different names for one or the whole. For example, we can superimpose two halves over the whole and

FIGURE 12.6
Representing fractions on grid paper.

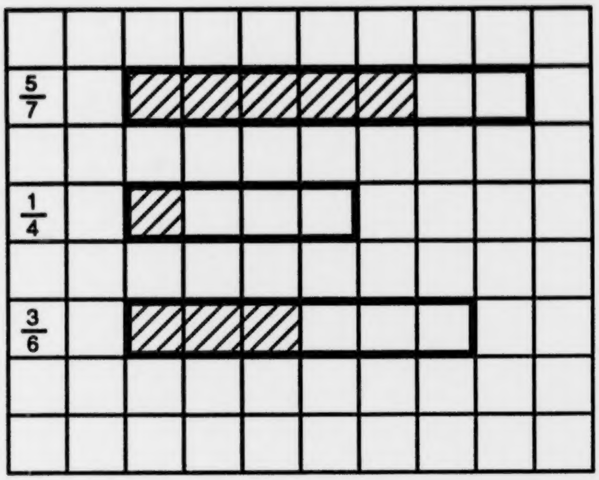
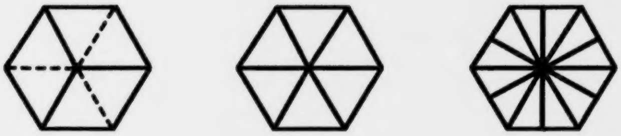


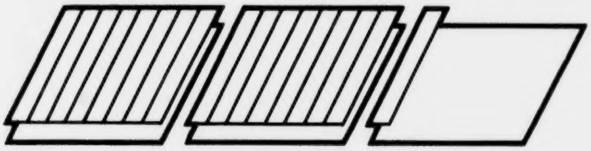
FIGURE 12.7
Hexagonal subdivisions.



point out to the students that two halves take up the same amount of space as the whole.

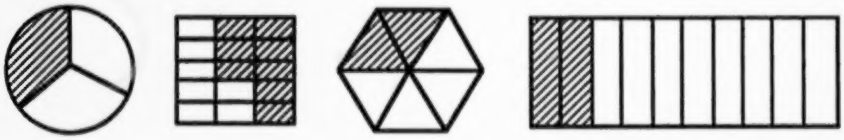
- Extend the above activities to demonstrate the meaning of *mixed numbers* and *improper fractions*. As in the example in Figure 12.8, use two or more sets of eighths to represent five-eighths, six-eighths, seven-eighths, eight-eighths (comparing them to the whole), nine-eighths, and so on. When you reach seventeen-eighths, exchange 16 of the pieces for 2 wholes, discuss the differences between the two models, and show how you can write the numerals for the two as $^{17}/_{8}$ or $2^{1}/_{8}$. Have students represent a variety of improper fractions and mixed numerals with the materials.

FIGURE 12.8
An activity for mixed numbers and improper fractions.



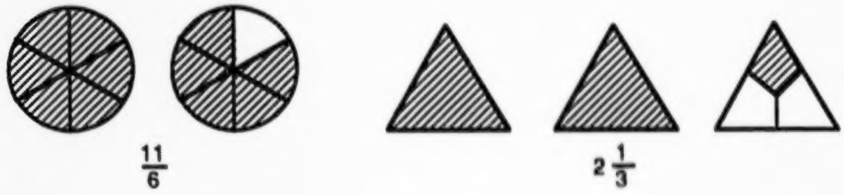
- If you have access to commercial fraction materials like Fraction Pieces and Fraction Bars, have the children represent a variety of fractions with the manipulatives and state the name of fractions represented by the shapes.
- Introduce the children to pictorial representations of fractions—shaded rectangles, squares, hexagons, circles, and/or other shapes that are divided into congruent parts and represent fractions, as are the examples in Figure 12.9. Ask them to state and write the fraction represented.

FIGURE 12.9
Pictorial representations of fractions.



Do similar activities with improper fractions and mixed numbers (see Figure 12.10). Have the students do many activities where they write the fraction associated with a shaded shape or a concrete representation of a fraction.

FIGURE 12.10
Pictorial representations of improper and mixed fractions.

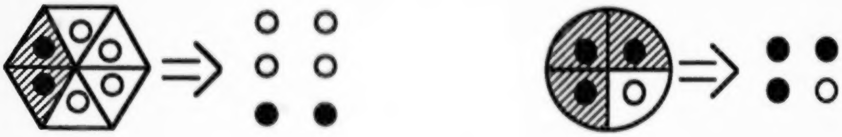


Activities for the Subset Meaning of Fractions

When working with a set of single objects, the number of objects in the whole group usually indicates the ***denominator*** of the fraction and the number of objects in the subgroup being considered is the ***numerator***. For example, if there are 27 people in a class, 7 of whom have blond hair, the fraction indicating the portion of the group having blond hair is $^{7}/_{27}$.

One way to introduce the ***set*** meaning of fractions is to begin with the part-whole meanings developed earlier. First, represent a fraction using a solid shape that has been appropriately subdivided, and then place single objects on the parts of the whole—one color for the portion of the whole being considered (the numerator) and another color for the remainder of the parts. Figure 12.11 shows two examples.

FIGURE 12.11
Associating part-whole and set meanings of fractions.



Next, consider these objects separately, and their relationship to the whole. When working with children who have a good understanding of the part-whole meaning of fractions, you may progress quickly from concrete to pictorial activities.

- Using red and white (or any two other colors) chips represent several different proper fractions and discuss the verbal names for the representation. For example, if you have a set of 5 red chips and 2 white chips, the red chips represent five-sevenths of the total set and the white chips represent two-sevenths of the set. Represent other fractions with chips or other materials (grains of popcorn and dried beans, two different colors of marbles, two different kinds of toys, etc.).

- Say different names of proper fractions and have students represent them with sets of objects.

- Draw pictures to represent fractions and have the student name the fraction. For example, draw nine solid circles and two clear circles, as shown in Figure 12.12, and ask the students to say what part of the whole set the solid dots represent. Do similar activities with other shapes or pictures like stars, people, trees, animals, or geometric shapes.

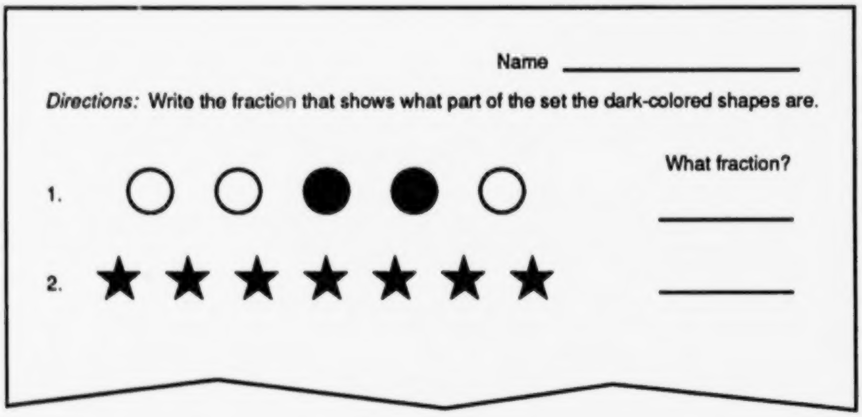
FIGURE 12.12
Using circles to represent fractions.

- Write fractions and have students draw pictures to represent them.

EXAMPLE: Draw **X**s and **O**s to show the fraction $5/6$.

- Show the student physical and pictorial representations of fractions from sets of discrete objects and have her write the corresponding fraction and vice versa. Include representations of "one" (five-fifths, etc.). See Figure 12.13 for sample worksheet problems.

FIGURE 12.13
A worksheet for the "set" meaning of fractions.



Name ______________

Directions: Write the fraction that shows what part of the set the dark-colored shapes are.

What fraction?

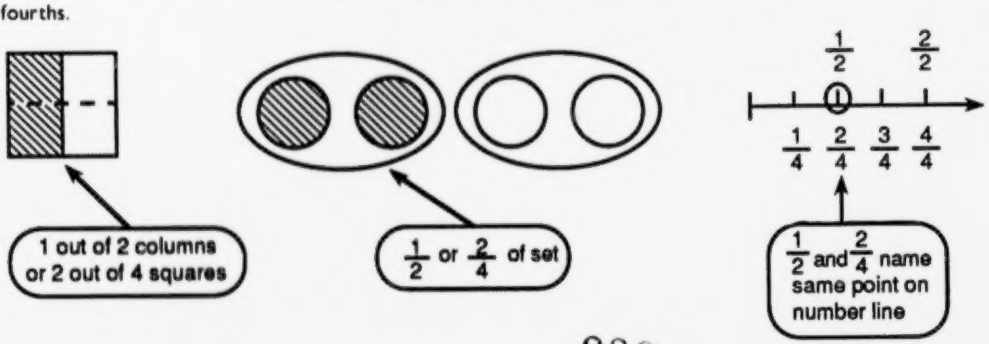
1. ○ ○ ● ● ○ ______

2. ★ ★ ★ ★ ★ ★ ★ ______

Activities for Equivalent Fractions (Objective C)

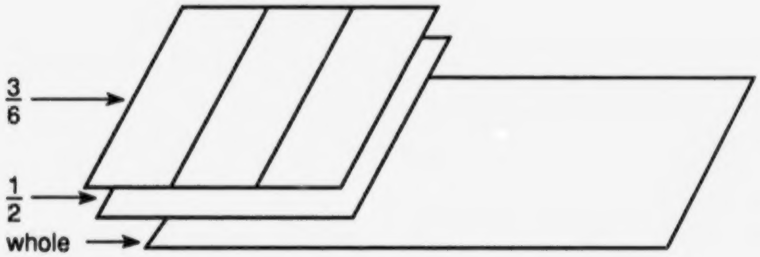
The major idea that needs to be communicated about equivalent fractions is that equivalent fractions are different names for the same number. One-half and two-fourths name the same part of a whole, the same part of a set, or the same point on a number line. Figure 12.14 illustrates models of this concept.

FIGURE 12.14
One-half and two-fourths.


1 out of 2 columns or 2 out of 4 squares
$\frac{1}{2}$ or $\frac{2}{4}$ of set
$\frac{1}{2}$
$\frac{2}{2}$
1
$\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$
$\frac{1}{2}$ and $\frac{2}{4}$ name same point on number line

To develop understanding of this idea using manipulatives, have students juxtapose cut-out pieces representing the two fractions to determine if they take up the same amount of space (see Figure 12.15).

FIGURE 12.15
Equivalent fractions.



■ Use arrays of chips or other objects to develop the idea of equivalent fractions. First, represent a fraction like $^{3}/_{4}$ using two colors of chips as in Figure 12.16. Next, add a row so that $^{6}/_{8}$ is represented, as in Figure 12.17.

FIGURE 12.16
A model for $^{3}/_{4}$.

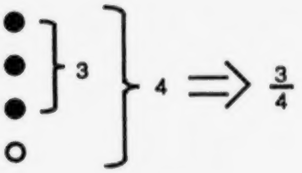
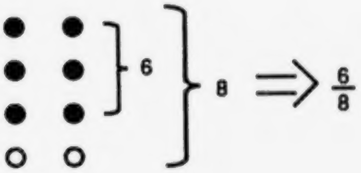
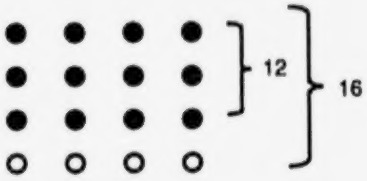


FIGURE 12.17
A model for $^{6}/_{8}$.



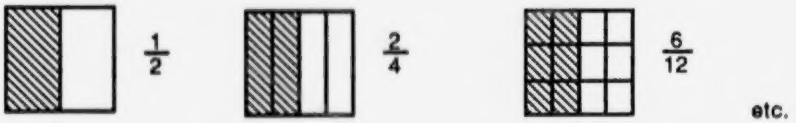
Continue to add rows, and discuss what fraction the array of chips names. (See Figure 12.18.) Finally, discuss the fact that adding new rows does not change the fact that 3 of the 4 rows are one color, and therefore that all the fractions you have represented are equivalent to $^{3}/_{4}$.

FIGURE 12.18
Showing equivalent fractions with arrays.



■ Use shaded shapes to represent equivalent fractions. For example, start with a representation of $^{1}/_{2}$, then subdivide each half into halves, and state or write the new fraction represented, as shown in Figure 12.19. Discuss the fact that the shaded area has not changed; you have simply changed the way the whole is subdivided.

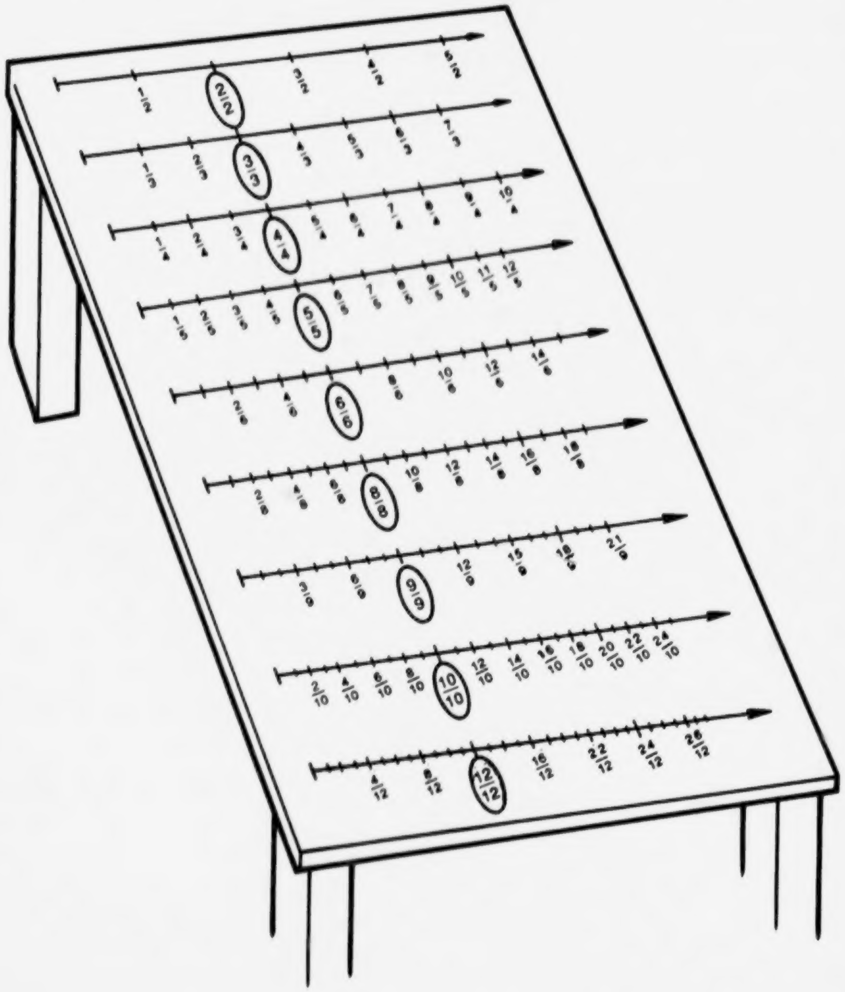
FIGURE 12.19
Showing equivalent fractions with shaded shapes.


1/2
2/4
6/12
etc.

- Show the students pairs of fractional numerals. Ask them to use materials to find out if they are equivalent, and if not, which is larger (i.e., which covers the most area).

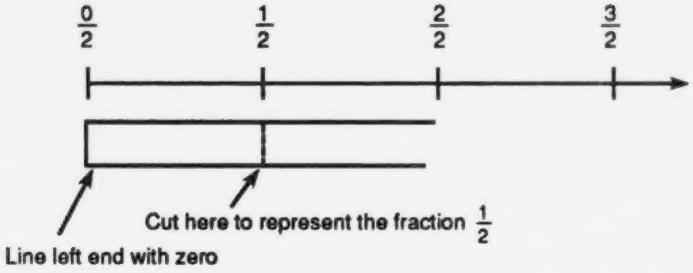
A *fraction table* (Wiebe 1985) is a useful way to investigate equivalent fractions and operations on fractions. Make 10 number lines from strips of paper, 2" wide and 2'6" long. Make number lines for wholes, halves, etc. (see Figure 12.20), so that 0 is at the left side of the strip, 1 is exactly one foot away, and so on. Each number line should go beyond 2 and should be marked for each fraction (e.g., the thirds number line should have marks for 0/3, 1/3, 2/3, 3/3, 4/3, 5/3, and 6/3). Laminate them and tape them to a table so that they are parallel to each other as in Fig. 12.20. (An alternative is to draw similar fraction number lines on a large poster.)

FIGURE 12.20
A "fraction" table.


1/2 2/2 3/2 4/2 5/2
1/3 2/3 3/3 4/3 5/3 6/3 7/3
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5 9/5 10/5 11/5 12/5
2/6 4/6 6/6 8/6 10/6 12/6 14/6
2/8 4/8 6/8 8/8 10/8 12/8 14/8 16/8 18/8
3/9 6/9 9/9 12/9 15/9 18/9 21/9
2/10 4/10 6/10 8/10 10/10 12/10 14/10 16/10 18/10 20/10 22/10 24/10
4/12 8/12 12/12 16/12 22/12 24/12 26/12

You also need to prepare strips of paper, as shown in Figure 12.21, about half an inch wide, and cut to equal the lengths of various fractions represented on the table, thus physically representing the fraction.

FIGURE 12.21
Fraction strips.



- Have the children use trial and error with the fraction table to find the fractions equivalent to $^1/_2$. To do this, have them place one end of the $^1/_2$ strip on the zero of one of the number lines and see if the other end falls exactly on a fraction on that number line. Tell them to keep trying different number lines until it does. When it does, they have found another name for the fraction and can record the equivalent fractions. Have them find and write fractions equivalent to a variety of proper, mixed and improper fractions.

At the symbolic level, it is necessary to develop the idea that a fraction, $^a/_b$, is equivalent to another fraction, $^c/_d$, if

$$\frac{a}{b} = \frac{c}{d} \times \frac{n}{n}$$

- Attempt to lead the student to discover that if you multiply the numerator and denominator of a fraction by the same number, your new fraction is equivalent to the original fraction. One way to do this is to write the fractions for several pairs of models or pictures of equivalent fractions, then investigate the relationship between the numerators and denominators in this series of fractional numerals.

- Have the student multiply several given fractions by the sequence $^1/_1$, $^2/_2$, $^3/_3$, $^4/_4$, $^5/_5$, etc. Draw pictures of the resulting fractions. Discuss the relationship between the pictures.

TEACHING DECIMALS

Models for Decimal Numerals (Objectives D–G)

One of the best models for developing meaning for decimal numerals is base-ten blocks. When using this manipulative for decimals, the large block becomes the unit, the "flat" becomes the tenth, the "long" the hundredth, and the small cube the thousandth. Figure 12.22 shows how you would represent 2.45 with base-ten blocks.

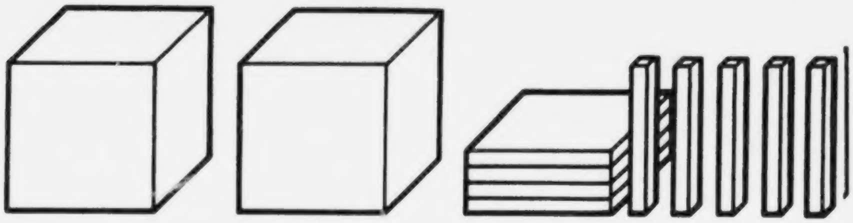
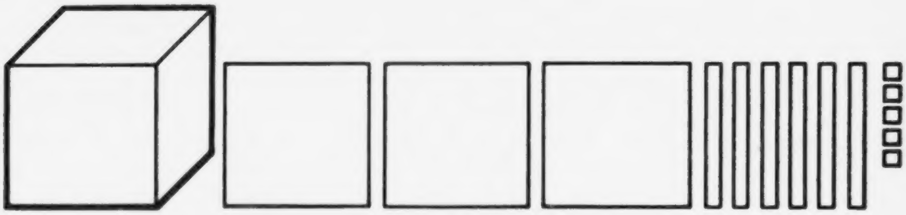


FIGURE 12.22
Base-ten block representation of 2.45.

A similar model, shown in Figure 12.23, uses a 10 cm by 10 cm square cut from tagboard or construction paper to represent units, a 10 by 1 rectangle for tenths, and a 1 by 1 square for hundredths.

FIGURE 12.23
A base-ten model.

Model for 1.365



Note: Cut the unit as a two dimensional shape ( ). Next, draw lines on it to give it a 3- dimensional appearance ( ).

- Have students represent a variety of decimals with the materials. Represent decimals with pictures or models and have students write the corresponding decimal numeral. Have students identify the place values in decimal numbers represented with the materials.

Pictorial activities for decimal numeration can be very similar to those discussed above for the part-whole meaning of fractions. The shapes students work with should be divided into tenths, hundredths, or thousandths. Activities should involve decimals in tenths, hundredths, thousandths, and so on, mixed decimals (ones that include units, tens, etc.), and decimals with fractions (e.g., .37 ½).

- Verbally give names of decimals to students and ask them to represent the numbers by shading shapes (predivided into ten or one-hundred equal squares).

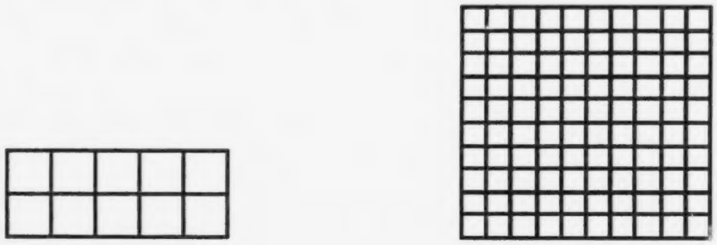


FIGURE 12.24
Pictorial activities for decimal numbers.

"Shade three-tenths of the shape."

"Shade 78 hundredths of this square."

Show the students how to express these fractional parts as decimal numerals.

$\frac{3}{10}$ = .3 (.3 means 3 × $\frac{1}{10}$—the 3 is in the *tenth's place*)

Do several more exercises of this sort.
Expand these examples to include hundredths and thousandths.

- Show the student pictures of decimal fractions, and ask him to write the corresponding decimal numerals. Early activities should involve only tenths and hundredths, but later activities can involve thousandths, ten-thousandths, and so on.

- Show the student how to use pictures to show the meaning of mixed decimal fractions, for example, 3.5 or 23.07. Then, have the student do exercises where she shades shapes to represent mixed decimal fractions or writes mixed decimal numerals to represent shaded shapes. (See Figure 12.25.) Include activities involving decimals with fractions (e.g., .3⅓). Have the student shade shapes to represent such decimals.

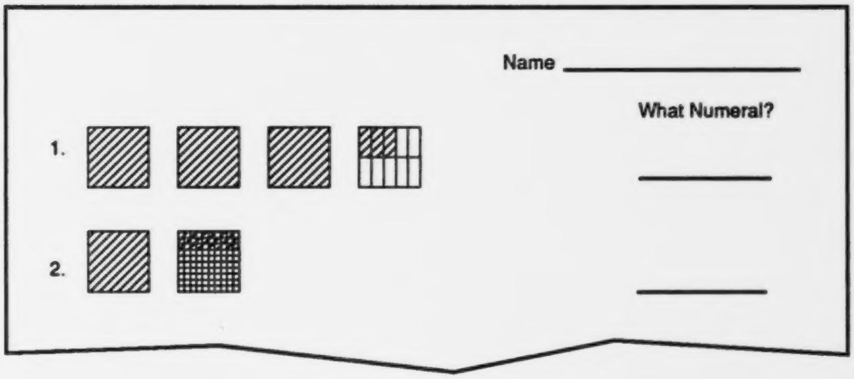
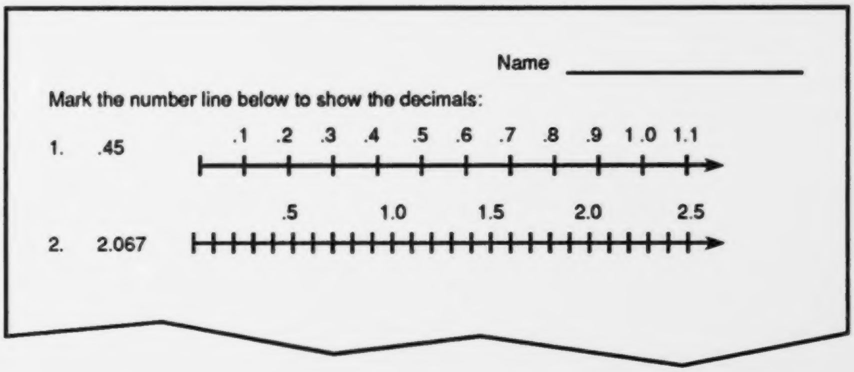


FIGURE 12.25
Pictorial worksheet for mixed decimals.

- Use number lines, as in Figure 12.26, to represent decimal numerals: represent decimals in tenths and hundredths on a number line marked in tenths; represent decimals in units, tenths, and hundredths and thousandths on number lines marked in units, tenths, and hundredths, and so forth.

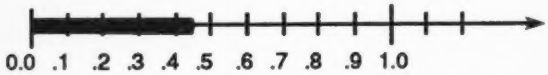


Mark the number line below to show the decimals:

FIGURE 12.26
A number line worksheet for decimals.

- Represent decimals on number lines and ask the student to write a decimal that corresponds to the picture. Ask her to name several more decimals that could also name the picture. For example, the picture in Figure 12.27 could represent the decimals .45, .451, or .45007.

FIGURE 12.27
Showing approximate decimals.



If the student names or writes a decimal that is unreasonable, graph that decimal for her to show how the picture differs from the one being considered.

Activities for Equivalent Decimals (Objective H)

Activities for equivalent decimals can involve shading and subdividing shapes and converting decimals to fractions. Shape-shading activities are similar to those described earlier with fractions. For example, to show that .3 = .30 = .300, shade three tenths of a square, draw additional dividing lines to show hundredths, then subdivide again to show thousandths. The important concept that needs to be communicated to the student is that the same portion of the whole is shaded or represented regardless of whether it is divided into tenths, hundredths, or thousandths.

- Ask the student to demonstrate the equivalence of .7, .70, and .700.

- At the symbolic level, convert equivalent decimals to fractions, and reduce the fractions. Students can then see that .5, .50, and .500 are equivalent because ${}^{5}/_{10} = {}^{50}/_{100} = {}^{500}/_{1000}$ (= ${}^{1}/_{2}$)

Symbolic Activities for Decimal Numeration (Objectives I–M)

Demonstrate expanded notation expressions for decimal numerals. Essentially, these activities are an extension of whole number activities introduced in Chapter 9.

Decimals can and should be represented in a variety of expanded notation formats:

1. Using the written place names:

   .85 = 8 tenths, 5 hundredths

2. Writing the sum of the total value in each place:

   1.35 = 1 + .3 + .05
   or = .05 + 1 + .3

3. Writing place values as fractions:

   $$.128 = 1 \times \frac{1}{10} + 2 \times \frac{1}{100} + 8 \times \frac{1}{1000}$$

4. Writing place values as decimals:

$$.407 = 4 \times .1 + 7 \times .001$$

5. Writing places as exponents:

$$3.245 = 3 \times 10^{0} + 2 \times 10^{-1} + 4 \times 10^{-2} + 5 \times 10^{-3}$$

Students should be able to convert decimal numerals to expanded notation in descending order of place value or in mixed order, as in example #2 above. They should also be able to translate expanded notation expressions to decimal numerals. Activities can also involve expanded notation expressions with missing parts that must be completed by the student.

EXAMPLE: Fill in the blanks in the problem below:

23.05 = ____ × .01 + 3 × ____ + ____ × 10

Activities for selecting the largest or smallest of several decimals can be related to pictorial experiences. A major understanding is that with decimals, adding zeros to the right of the non-zero digits does not change its value, but adding zeros between the decimal point and the first non-zero digit does (e.g., .3 is different from .03 but not from .30). This is opposite to what happens with whole numbers. Failure to realize this may cause students great difficulty when comparing the relative size of decimal numerals.

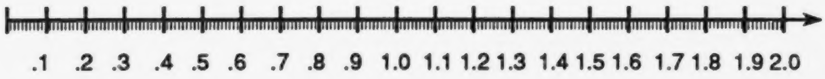
Rounding Decimals (Objective L)

Discuss the meaning and importance of rounding. For example, if the population of the United States is 254,894,345 it is not important to say this entire number every time we talk about the population of the United States, so we round to 250 million. The same is true for small numbers involving decimals. If a certain professional baseball player got 125 hits out of 453 at bat, his batting average would be .2759381. . . , but instead of saying this number to eight or ten decimal places, we round it to the nearest thousandth and say he hit .276. A calculator with an eight-digit display would be a useful aid in this discussion.

Probably the most useful model for teaching students about rounding decimals is the number line, illustrated in Figure 12.28. When rounding to the nearest tenth, have the student locate the decimal on a number line marked in tenths, then decide which tenth it is closest to.

FIGURE 12.28
Rounding decimals.

Round 1.67 to the nearest tenth



At the symbolic level, the problem "Round .067 to the nearest hundredth" can be translated to "Is .067 closer to .06 or .07?" This problem can be solved by subtracting .06 from .067 and .067 from .07. This would be a good place to use a calculator. The smallest difference tells you which number to round to.

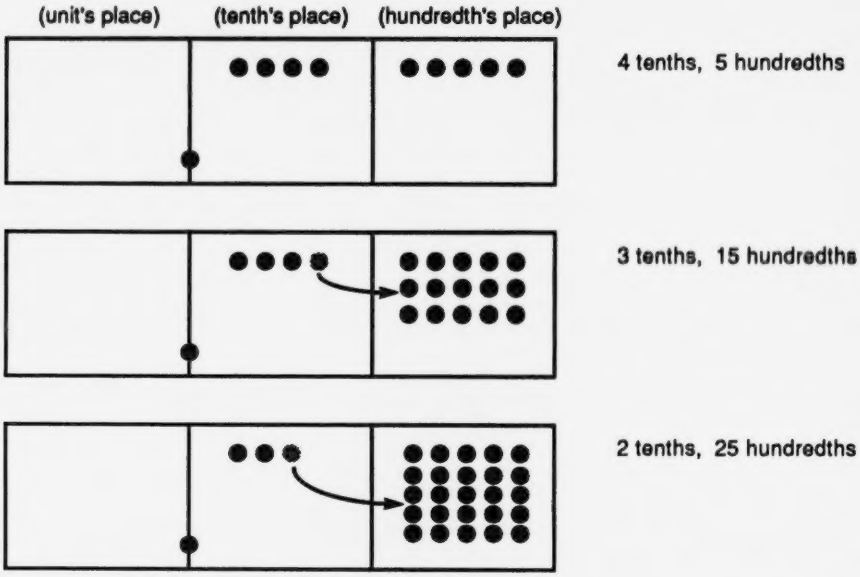
The algorithm for rounding decimals is similar to that for rounding whole numbers. In rounding to the nearest tenth, for example, if the digit immediately to the right is five or larger, increase the tenths digit by one (and delete all digits to the right); otherwise, do not change that digit and drop all digits to the right. Children should be given a variety of rounding activities including rounding numbers like 253.45 to the nearest *ten*.

Regrouping Decimals (Objective M)

Regrouping activities can start with concrete materials or pictures. The place-value charts in Figure 12.29 show three different representations for .45. The first picture shows the usual representation of .45. In the second, a tenth has been exchanged for 10 hundredths. In the last, another tenth has been exchanged for 10 hundredths. The total value represented in each picture, however, is the same.

Next, explain how to regroup symbolically and give the student practice with a variety of regroupings at the symbolic level.

FIGURE 12.29
Place-value charts for decimals.



EXAMPLE: Fill in the blanks below:

1. .456 = 4 tenths + 4 hundredths + ____ thousandths
2. 34.55 = 3 tens + 3 ones + ____ tenths + ____ ones

Converting between Decimals and Fractions (Objective N)

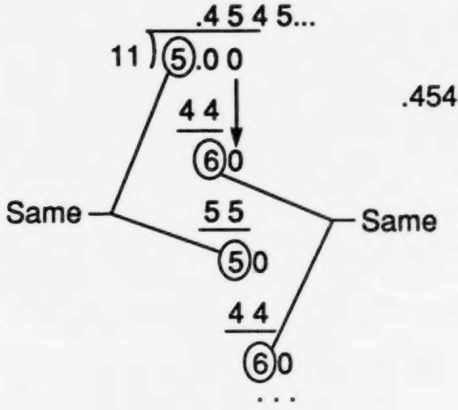
Beginning activities for conversion from decimals to fractions, and vice versa, should involve denominators that are factors of a power of ten. For example, $^4/_5$ can be converted to a decimal by finding an equivalent fraction whose denominator is 10, 100, 1000 or another power of ten: $^4/_5$ = $^8/_{10}$ = .8.

- Give students a variety of fractions where the denominator is a factor of 10, 100, or 1000. Include mixed fractions. Ask them to convert these fractions to decimals.

When the denominator is not a factor of a power of 10, show students that they can convert to decimals by dividing the numerator of the fraction by the denominator. (See Figure 12.30.) Previous experience with equivalent decimals will prepare the student to understand the practice of annexing zeros and carrying the division to the right of the decimal. Instruction should involve a discussion of repeating decimals, when they occur, and how they are represented. Once we get a remainder that is the same as an earlier remainder, a repeating pattern has been established. (Note: The repeating pattern is not necessarily established when a digit appears for a second time in the quotient.)

FIGURE 12.30
Converting from fraction to decimal.

Example: Change $\frac{5}{11}$ to a decimal.



$.454545... = .\overline{45}$

Converting from a decimal to a fraction involves recognizing the place value of the last digit in the decimal numeral, writing this value as the denominator, then reducing the resulting fraction.

EXAMPLE: 4.45—the decimal is "four and forty-five hundredths." Thus the denominator becomes 100 and we have $^{445}/_{100}$ (or $4\,^{45}/_{100}$).

A problem occurs when we have a repeating decimal. Many students, when asked to convert a decimal like .333333 . . . ($.\overline{3}$) to a fraction, simply truncate and write $^{333333}/_{1000000}$. Actually, the fraction equivalent to .333333 . . . . is $^{1}/_{3}$. Fortunately, there is an algorithm for converting repeating decimals to fractions. It is shown in Figure 12.31.

FIGURE 12.31
An algorithm for converting repeating decimals to fractions.

Example 1: Convert $.1212\overline{12}$ to a fraction.

Step 1: $x = .1212\overline{12}$ Set the decimal equal to x.

Step 2: $100x = 12.1212\overline{12}$ Multiply both sides of the equation by the power of 10 needed to move the decimal to the right of the first block of digits that repeats. In this example, we multiply by 100 since we want to move the decimal 2 places to the right. (*Note:* We want to isolate the first block of digits as a whole number and line up all corresponding digits so that when we subtract, all digits to the right of the decimal become zero.)

Step 3:

$$\begin{array}{r} 100x = 12.1212\overline{12} \\ \underline{-\ x = .1212\overline{12}} \\ 99x = 12.000000 \end{array}$$

Subtract the first equation from the second.

Step 4: $x = \dfrac{12}{99}$ Solve for x.

Step 5: $x = \dfrac{4}{33}$ Reduce, if possible.

Example 2: Convert $.1666\overline{6}$ to a fraction.

Step 1: $x = .1666\overline{6}$

Step 2: $100x = 16.666\overline{6}$

Step 3:

$$\begin{array}{r} 100x = 16.666\overline{6} \\ \underline{-\ x = .166\overline{6}} \\ 99x = 16.5 \end{array}$$

Step 4: $x = \dfrac{16.5}{99}$

Step 5: $x = \dfrac{16.5}{99} = \dfrac{165}{990} = \dfrac{1}{6}$

In the age of technology, we all should have the decimal equivalents of commonly-used fractions on the tip of our tongues. For example, if we want to use the calculator to add $12\frac{1}{2}$ inches to $6\frac{1}{4}$ inches, it is much easier to enter

12.5 + 6.25 =

than 1 ÷ 2 + 12 + M+ 1 ÷ 4 + 6 = + MR =

Other operations involving fractions can be even more difficult. Knowing the decimal equivalents of common fractions makes doing operations involving measurement much easier.

At a minimum, children should know the decimal equivalents of the following fractions, and vice versa: $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{7}{8}$. From these, they should readily be able to find the decimal equivalents of numbers like $1\frac{3}{8}$ and $5\frac{1}{3}$.

- Give students practice in these basic facts as you do the other basic math facts. Use games, timed drills, and so on. For suggested activities, see Chapter 20.

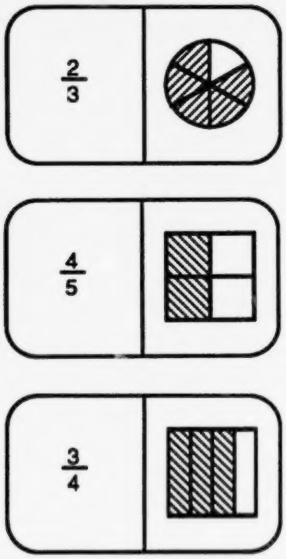
SAMPLE DRILL AND PRACTICE ACTIVITIES

Activity 12.1: Fraction Dominoes

Objective: Associate a numeral with a picture of a fraction.

Materials: Fraction Dominoes—30 or so dominoes made from tagboard where one side of the domino contains a fractional numeral and the other a picture of a fraction, as shown in Figure 12.32. The following fractions may be arranged randomly on the dominoes so that each fraction and picture appears five times: $^2/_3$, $^2/_4$, $^3/_4$, $^3/_5$, $^4/_5$, $^5/_6$. Include six doubles (same numeral and picture on a domino—one for each fraction).

FIGURE 12.32
Fraction dominoes.



Number of Players: 2 or 3

Methods:

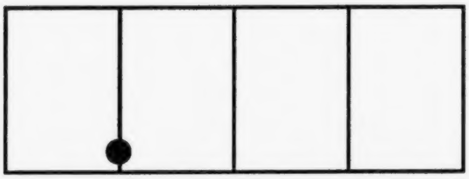
1. The dominoes are placed face down in the center of the playing area and shuffled.
2. Each player draws a domino to see who starts—the person with the largest fraction (either numeral or picture) starts.
3. The dominoes are replaced and reshuffled and each player draws 7 dominoes.
4. The person starting places a domino on the table. The next player may play a domino to match either end of the first one played—either picture to picture or numeral to picture.
5. A "double" is played crosswise, allowing further play at either end. Note that a domino may be played only at the unmatched end of another domino.
6. If a player cannot play, he or she draws from the pile until a match is found. If the pile has already been distributed, a player passes when he cannot make any matches.
7. The game is over when one player has played all his dominoes and thus is the winner.

Activity 12.2: Decimal Bankers Game

Objective: To regroup at the concrete level in the tenths, hundredths, and thousandths place.

Materials and Methods: Similar to the Chip Trading Game (Chapter 9) except that the playing board is a four-place place-value chart with a decimal point between the one's and tenth's places. (See Figure 12.33.) Each side of the die represents a certain number of thousandths. (⚄ = .005.)

FIGURE 12.33
A playing board for Decimal Bankers Game.



The winner is the first person to obtain a marker in the ones' place.

Activity 12.3: Equivalent Pairs

Objective: Recognize equivalent pairs of fractions.

Materials: A set of 12 to 15 pairs of playing cards and a "spider" card, as in the examples in Figure 12.34. Each card in a pair has one of two equivalent fractions (e.g., $^1/_2$ and $^2/_4$). The spider card has the picture of an ugly spider.

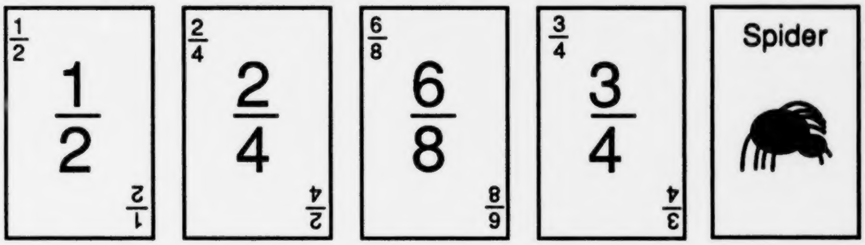


FIGURE 12.34
Playing cards for equivalent pairs.

Number of Players: 3 to 5

Methods:

1. The cards are shuffled and dealt to the players.
2. Players remove all pairs of equivalent fractions they have in their hands and place them face up on the table in front of them.
3. The person to the left of the dealer draws one card from the dealer's hand (without looking at the dealer's cards) and places it in his hand. If a pair is formed with this card, the pair is placed on the table.
4. The next person to the left removes one card from the hand of the person who just played, etc. Play continues, moving in a clockwise direction until someone has no more cards in his or her hand.
5. At this point, the person with the most pairs is the winner and the person with the spider card must "kiss the spider."

SUGGESTED READINGS

Arithmetic Teacher 31(6), February 1984. The entire issue is dedicated to rational numbers.

Beede, R. B. Dot method for renaming fractions. *Arithmetic Teacher* 33(2), October 1985: 44-45.

Brown, C. N. Fractions on grid paper. *Arithmetic Teacher* 27(5), January 1979: 8-10.

Chiosi, L. Fractions revisited. *Arithmetic Teacher* 31(8), April 1984: 46-47.

Kalman, D. Up fractions! Up $^n/_m$! *Arithmetic Teacher* 32(8), April 1985: 42-43.

Swart, W. L. Fractions vs. decimals—the wrong issue. *Arithmetic Teacher* 29(2), October 1981: 17-18.

Usiskin, A. P. The future of fractions. *Arithmetic Teacher* 27(5), January 1979: 18-20.

Van de Walle, J., and Thompson, C. S. Fractions with counters. *Arithmetic Teacher* 28(2), October 1980: 6-11.

Wiebe, J. H. Discovering fractions on a "fraction table." *Arithmetic Teacher* 33(4), December 1985: 49-51.

QUESTIONS

1. Obtain a current newspaper and record the number of whole numbers, negative integers, fractions, decimals, and percentages used in the paper, excluding the stock-market report. What conclusions can you make regarding the need to teach the meaning of different kinds of numerals (and numbers) in school?
2. Read Swart's and Usiskin's articles (see Suggested Readings) and discuss the place of fractions in the curriculum in light of the presence of calculators and computers in our society.
3. Poll a group of working adults and ask them which kinds of numbers they have used in the past month—whole numbers, negative numbers, fractions, decimals, and percentages. Also ask if they have added, subtracted, multiplied, or divided any of these types of numbers. What do you conclude?
4. Write a computer program that will take input in the form of a fraction and produce output of the decimal equivalent. Repeat with a fraction input and percentage output. Advanced activity: write a program that accepts input as a fraction, decimal, or percent and converts the number to either of the other two forms, depending on the request of the user.

CHAPTER

13 Operations on Fractions

Although it is very important that children be taught the meaning of fractional concepts and numerals, one can question the value of learning to add, subtract, multiply, and divide fractions in the traditional way. Outside of measurement, real-life situations calling for operations on fractions rarely occur, and, as we slowly move toward the metric system, such operations will become even rarer. Instead of spending a great deal of time teaching operations on fractions, we should be de-emphasizing them and spending more time on operations on decimals and percentages.

Another reason why operations on fractions using traditional techniques will become less important in the future is that calculators and computers deal much more easily with decimal numerals than with fractions. One useful way to solve any operation on fractions is to simply change the operands to decimals. Then, if need be (e.g., when you want a length to be expressed in traditional units like 1⅞ inch), change the answer back to a fraction. Even if we continue to teach operations on fractions, we should also teach children to use calculators to solve problems involving fractions by converting the numbers in the problems to decimals, then operating on the decimals.

Having made a case for de-emphasizing operations on fractions, we will proceed with suggestions for teaching such operations: virtually all district and state curriculum guidelines and standardized tests for mathematics have major components involving operations on fractions. In addition, until the United States has switched to the metric system, some knowledge of operations on fractions will continue to be important. It is the author's opinion that the major emphasis in fractions should be on the ideas from the previous chapter, and on using calculators and computers to solve problems with fractions, but that students should be given some instruction on the skills in this chapter before or after they have been taught to add, subtract, multiply, and divide decimals.

TEACHING OPERATIONS ON FRACTIONS

Many of the computational algorithms involving fractions are deceptively simple. It seems easy enough to remember rules like "multiply numerator by numerator and denominator by denominator" or "invert and multiply." The fact is, though, that many students have a difficult time with operations on fractions. In order for students to develop proficiency with these operations, teachers need to base instruction on increasing understanding of basic fraction concepts and of the algorithms for operations on fractions. Without true understanding, the rules for operations on fractions are very confusing: in some of the operations you find common denominators, in others you do not; in some you operate on numerators only, in others you operate on numerators and denominators.

This chapter emphasizes activities for developing meaning for operations on fractions so that the rules for manipulating them will be easier to remember. Note that instruction in operations with fractions should not begin until the student has mastered the meanings of fractions presented in Chapter 12.

Behavioral Objectives for Operations on Fractions

A. Find the sum of two fractions (with like or unlike denominators) using concrete materials.
B. Find the difference between two fractions (with like or unlike denominators) using concrete materials.
C. Find the sum of two proper fractions with like denominators.
D. Find the difference of two proper fractions with like denominators.
E. Find the product of two proper fractions.
F. Find the quotient of two proper fractions.
G. Find the sum of two proper fractions, improper fractions, or mixed numerals with like or unlike denominators.
H. Find the difference between two proper fractions, improper fractions, or mixed numerals with like or unlike denominators.
I. Multiply proper, improper, or mixed fractions.
K. Solve verbal problems involving operations on fractions.
L. Estimate the sum, difference, product, or quotient of two fractions.
M. Use a calculator to add, subtract, multiply, or divide fractions by conversion to decimals.
N. Use the PRINT statement in Logo or BASIC to add, subtract, multiply or divide fractions.

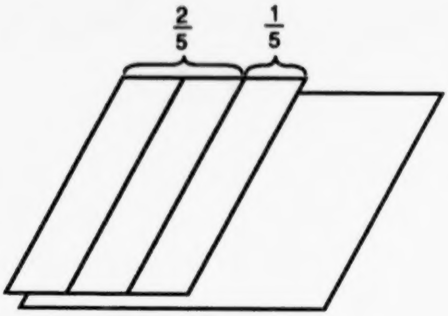
Manipulative and Pictorial Activities for Adding and Subtracting Fractions (Objectives A, B)

Manipulative activities for operations on fractions—an extension of those introduced in Chapter 12—involve shapes and number lines. Using number lines and shapes it is possible for students to find sums and differences of fractions with unlike denominators before having the ability to find least common denominators. Such experiences develop understanding of the operations on fractions and prepare them for success with symbolic manipulations.

- Purchase or make a set of fraction shapes. The unit or "whole" could be a 20 cm by 20 cm square of white tagboard. The halves would be 10 cm by 20 cm rectangles made of red tagboard. Thirds could be 6.67 cm by 20 cm from green tagboard, and so on. A minimum set of materials would contain a unit, halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths.

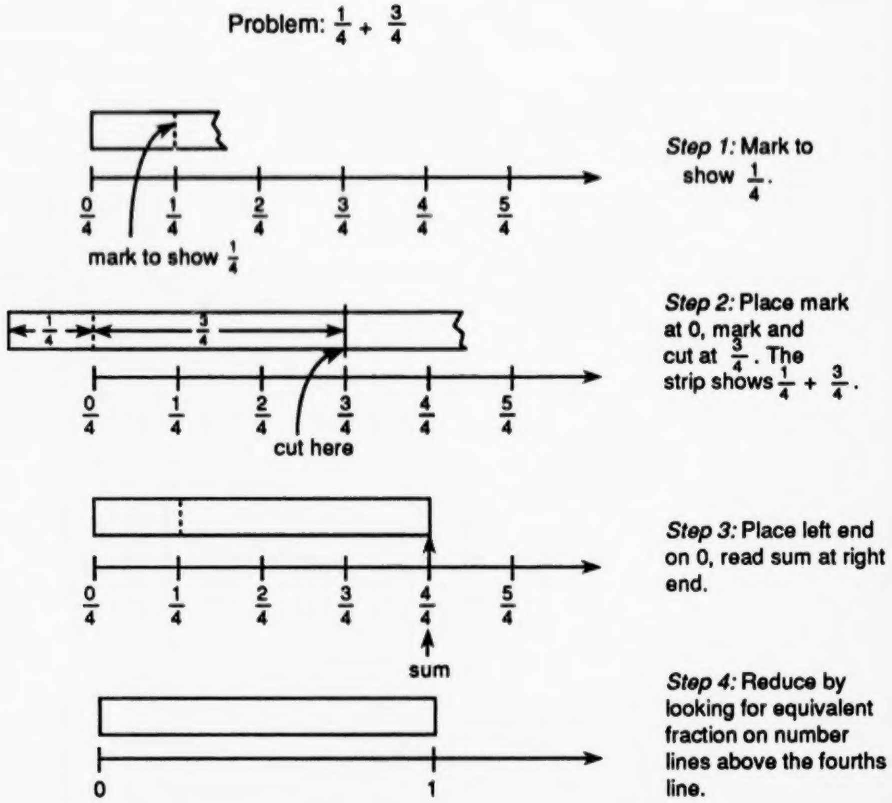
- Demonstrate how to add fractions with the shapes. Initially problems may be presented verbally and later they may be presented as number sentences. Start with fractions having like denominators. To find the sum of two fifths and one fifth, place two fifths on the unit, then place one fifth next to them (as in Figure 13.1) and state the answer. Later, write the answer.

FIGURE 13.1
Adding fractions with manipulatives.



Introduce the use of the fraction table (see Figure 12.20) for finding sums of fractions with like denominators. This procedure is outlined in Figure 13.2. To add two fractions on the fraction table, place the left end of a ½" strip of paper on zero and mark the strip for the first fraction. Next, place that mark on zero and mark for the second fraction. Cut the strip at the second mark. This strip represents the

FIGURE 13.2
Sums on the fraction table.


Problem: 1/4 + 3/4
0/4
1/4
2/4
3/4
4/4
5/4
mark to show 1/4
Step 1: Mark to show 1/4.
1/4
3/4
cut here
Step 2: Place mark at 0, mark and cut at 3/4. The strip shows 1/4 + 3/4.
sum
Step 3: Place left end on 0, read sum at right end.
0
1
Step 4: Reduce by looking for equivalent fraction on number lines above the fourths line.

sum of the two fractions. The numeral representing the sum may be found by placing the left end at zero and reading the fraction at the right end of the strip. The child may also experiment with the strip to find other names for the sum (see Chapter 12 for a discussion of equivalent fractions on the fraction table).

- Present a variety of exercises involving addition of fractions and ask the students to use the materials to solve the problem. Include problems with answers larger than one. When using the fraction table, have them experiment to see if they can reduce the answer (have them look for equivalent fractions only above the number line where they found their sum).

When using materials to subtract fractions with common denominators, as in Figure 13.3, first represent the minuend, then remove the amount indicated by the subtrahend and state or write the difference.

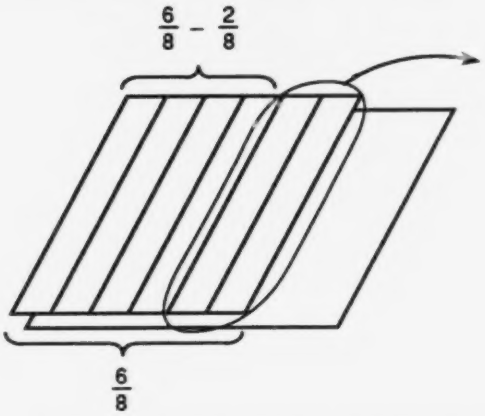


FIGURE 13.3
Subtracting fractions with like denominators.

Figure 13.4 shows how to subtract using the fraction table. First mark and cut a strip to represent the minuend. Next, subtract or remove the amount indicated by the subtrahend from the strip representing the minuend by marking the second fraction and cutting it off. The remaining piece is the difference.

- Ask students to do a variety of subtraction problems with the materials. First, give them a number of problems verbally, have them do the manipulations, then use the overhead to present the correct solution and discuss. Next, give them worksheets with written subtraction problems with a variety of like denominators for which you have materials. Have them use the materials to find answers, writing them on the worksheet.

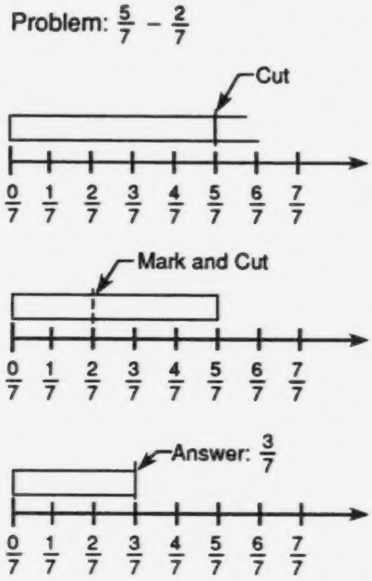
The algorithms for adding and subtracting fractions with like denominators should be directly related to the above manipulations. Solveproblems like $^{1}/_{4} + {}^{3}/_{4}$ with materials and write and discuss the number sentence representing the problem. Discuss why we add numerators and not denominators, using the materials to demonstrate. Present symbolic subtraction of fractions similarly. Have students do practice activities with materials available if difficulty arises.

Multiplying and Dividing Fractions (Objectives E, F)

Models and Manipulations

Models for multiplication of fractions range from the cut-out pieces discussed above to the fraction table. Instruction should be oriented toward developing un-

FIGURE 13.4
Subtracting on the fraction table.


Problem: $\frac{5}{7} - \frac{2}{7}$
Cut
Mark and Cut
Answer: $\frac{3}{7}$
$\frac{0}{7}$ $\frac{1}{7}$ $\frac{2}{7}$ $\frac{3}{7}$ $\frac{4}{7}$ $\frac{5}{7}$ $\frac{6}{7}$ $\frac{7}{7}$

Step 1: Mark and cut to represent $\frac{5}{7}$.

Step 2: From this strip, cut and remove $\frac{2}{7}$.

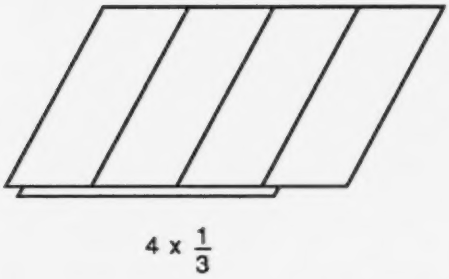
Step 3: Find answer.

Step 4: Reduce, if possible (check for equivalent fractions above sevenths line).

derstanding of what it means to multiply fractions and what answer to expect after the operation is completed.

- Start by using materials to model the product of a whole number and a fraction. Four times one-third would be modeled by placing four thirds side by side, as shown in Figure 13.5.

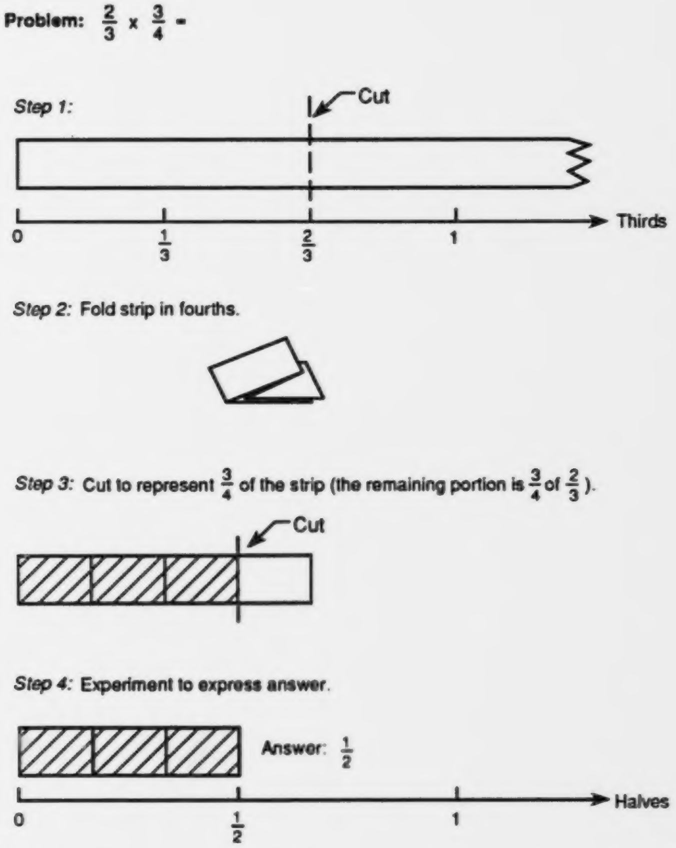
FIGURE 13.5
Modeling $4 \times {}^1/_3$.


$4 \times \frac{1}{3}$

- Have the student use the fraction table to find products of proper fractions and whole numbers. Use the same techniques used for finding sums of fractions (e.g., ${}^3/_4 \times 2 = {}^3/_4 + {}^3/_4$).

Demonstrate the use of the fraction table to find products of proper fractions. An example is given in Figure 13.6. To multiply one fraction by another using the fraction table, cut the strip of paper to represent the first fraction. Now, you want

FIGURE 13.6
Multiplying using fraction strips.


Problem: $\frac{2}{3} \times \frac{3}{4} =$
Step 1:
Cut
0
$\frac{1}{3}$
$\frac{2}{3}$
1
Thirds
Step 2: Fold strip in fourths.
Step 3: Cut to represent $\frac{3}{4}$ of the strip (the remaining portion is $\frac{3}{4}$ of $\frac{2}{3}$).
Cut
Step 4: Experiment to express answer.
Answer: $\frac{1}{2}$
0
$\frac{1}{2}$
1
Halves

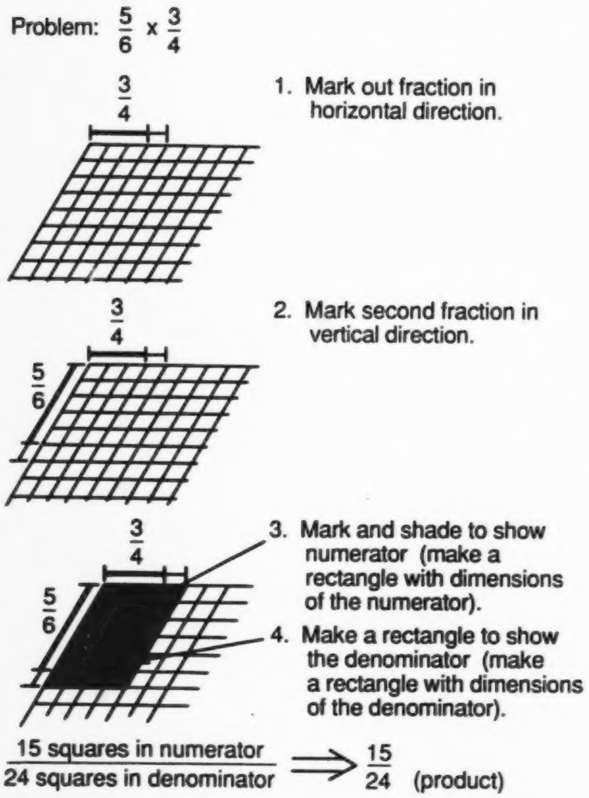
to find the portion of the strip represented by the second fraction. For example, in the problem $^2/_3 \times {}^3/_4$, you want to find three-fourths of a strip two-thirds units long. Therefore, you fold the strip into the number of equal parts indicated by the denominator of the second fraction and cut it to indicate the second fraction. Experiment with the fraction table to find a way to express the answer.

Another model is closely associated with the standard symbolic algorithm for multiplying fractions (where you multiply numerator by numerator and denominator by denominator). It involves arrays drawn and shaded on grid paper where each dimension represents one of the factors (see Figure 13.7). First, mark the horizontal direction on the grid to show the numerator and denominator of one fraction. Next, mark the vertical direction to show the numerator and denominator of the second fraction. Now, make and shade a rectangle to show the product of the numerators. Mark a rectangle to show the product of the denominators. The number of squares in the shaded rectangle is the numerator of the answer and the number of squares in the larger rectangle (including those in the shaded area) is the denominator.

- Give students a variety of problems involving multiplication of fractions and ask them to use the materials to find the products.

Models for dividing fractions are based on those used for developing understanding of division of whole numbers. The problem 24 ÷ 6 can be thought of as

FIGURE 13.7
Multiplying fractions using rectangles.



1. Mark out fraction in horizontal direction.
2. Mark second fraction in vertical direction.
3. Mark and shade to show numerator (make a rectangle with dimensions of the numerator).
4. Make a rectangle to show the denominator (make a rectangle with dimensions of the denominator).

$\frac{\text{15 squares in numerator}}{\text{24 squares in denominator}} \Rightarrow \frac{15}{24}$ (product)

either (1) the size of each jump if you start at 24 and go back to zero in six equal-sized jumps (partitioning), or (2) the number of equal-sized jumps, six units long, going from 24 to zero (measurement). When modeling division of a fraction by a whole number as in $3/4 \div 3$, it is best to use the partitioning model. Using the fraction table, cut a strip of paper $3/4$ of a unit long then divide it into 3 equal parts by folding (see Figure 13.8)—this is similar to taking three equal-sized jumps. Find the length of one of these sections by experimenting on other number lines.

When modeling division of a whole number by a fraction or a fraction by a fraction, it is best to use the measurement model. With this technique, the first fraction (the dividend) is the starting point on the number line. The second fraction (divisor) is the size of the jumps or the amount subtracted repeatedly from the first. The answer (quotient) is the number of equal-sized jumps (or the fraction of a jump) to get to zero.

Figure 13.9 provides an example with the problem "$3/4 \div 1/2 = ?$." With the model, we try to find how many times we can subtract $1/2$ from $3/4$. Start at $3/4$ and use strips of paper cut in half units. How many of the strips does it take to get to zero? The answer is $1\,1/2$.

Note that with this technique it is not always easy to obtain an exact answer. For example in the problem $3/4 \div 2/3$ in Figure 13.10, you use an entire strip of $2/3$ and a small part of another (answer: one plus a fraction). By folding, a few students might be able to determine that the small part of a strip is $1/8$ of a strip (therefore the answer is $9/8$), but most would say that it is approximately $1/5$, $1/6$, or another similar fraction. *The important point here is not to try to get chil-*

FIGURE 13.8
Dividing a fraction by a whole number using fraction strips.

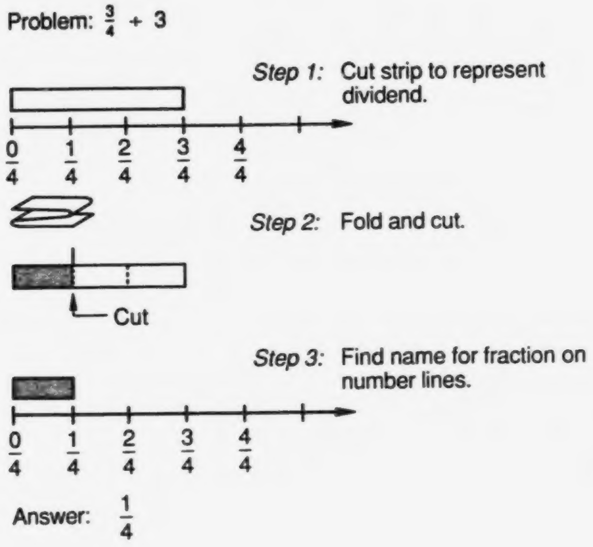
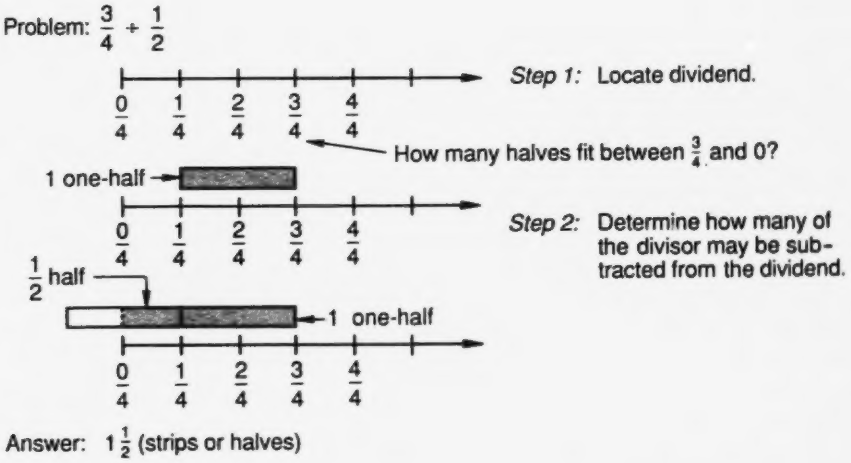

Problem: $\frac{3}{4} \div 3$
Step 1: Cut strip to represent dividend.
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$
Step 2: Fold and cut.
Cut
Step 3: Find name for fraction on number lines.
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$
Answer: $\frac{1}{4}$

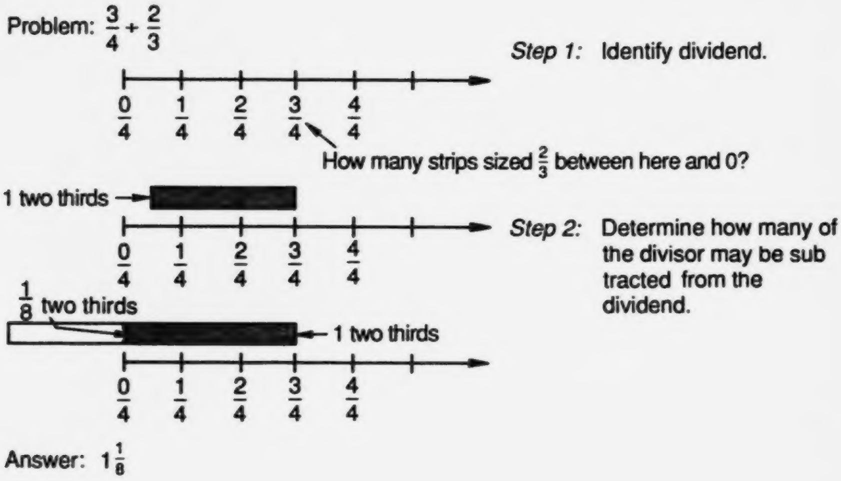
FIGURE 13.9
Dividing fractions using fraction strips.


Problem: $\frac{3}{4} \div \frac{1}{2}$
Step 1: Locate dividend.
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$
How many halves fit between $\frac{3}{4}$ and 0?
1 one-half
Step 2: Determine how many of the divisor may be subtracted from the dividend.
$\frac{1}{2}$ half
1 one-half
Answer: $1\frac{1}{2}$ (strips or halves)

dren to obtain exact answers to difficult problems involving division of fractions but rather to help them develop understanding of the process and concepts involved. Thus, if the student can use the materials to give a close approximation of the answer (in the above problem, $1\frac{1}{6}$ would be a good approximation), you have achieved your goal. *In general, when using this model, stick with simple problems that have answers containing halves, thirds, or fourths* (e.g., $\frac{3}{4} \div \frac{1}{2}$); then when understanding has been developed, move to the symbolic level.

With the fraction strips we can also divide fractions where the second fraction is larger than the first. In the problem $\frac{1}{2} \div \frac{2}{3}$ in Figure 13.11, for example, we cannot subtract the entire amount of two-thirds from one-half, even once. Our answer is therefore less than one—a proper fraction. The quotient is the part of the strip between the $\frac{1}{2}$ and the zero on the number line. Students may be able to determine the fraction of the strip between those two points by folding the strips, or

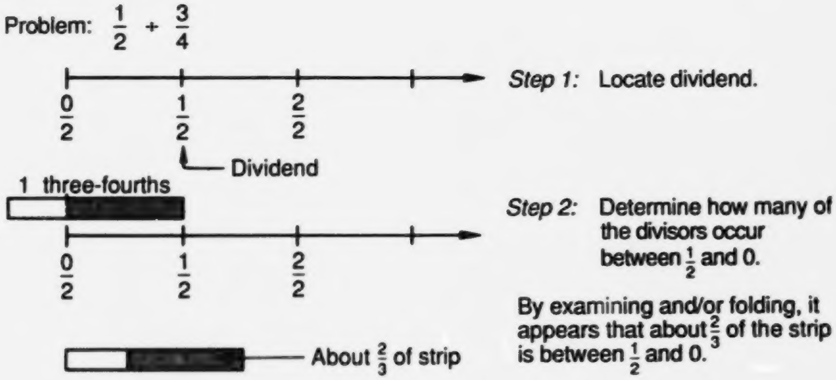
FIGURE 13.10
Inexact answer.


Problem: $\frac{3}{4} + \frac{2}{3}$
$\frac{0}{4}$
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
How many strips sized $\frac{2}{3}$ between here and 0?
1 two thirds
$\frac{1}{8}$ two thirds
1 two thirds
Answer: $1\frac{1}{8}$

Step 1: Identify dividend.

Step 2: Determine how many of the divisor may be sub tracted from the dividend.

FIGURE 13.11
Dividing when the second fraction is larger than the first.


Problem: $\frac{1}{2} + \frac{3}{4}$
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
Dividend
1 three-fourths
About $\frac{2}{3}$ of strip

Step 1: Locate dividend.

Step 2: Determine how many of the divisors occur between $\frac{1}{2}$ and 0.

By examining and/or folding, it appears that about $\frac{2}{3}$ of the strip is between $\frac{1}{2}$ and 0.

they may estimate the answer by shading the portion between ½ and 0 then simply looking at it.

After the student has participated in numerous experiences like those above, he should be able to give reasonable estimates of addition, subtraction, multiplication, or division of fraction problems. You have now won the war in your efforts to teach operation on fractions. If students have understood concrete manipulations, symbolic understanding will be much easier to achieve.

Symbolic Operations

The first step in developing proficiency with symbolic multiplication of fractions is to relate manipulative experiences to the algorithm. Using the grid model in Figure 13.7, the student should understand that the shaded square represents the numerator, and that its value (area) is the product of the two numerators. The value of the denominator is the product of the two denominators (see Figure 13.12).

■ After relating the model to the algorithm, give students sufficient practice. Again, if difficulties persist, allow students to use the materials to help find answers.

Problem: $\frac{3}{4} \times \frac{2}{5}$

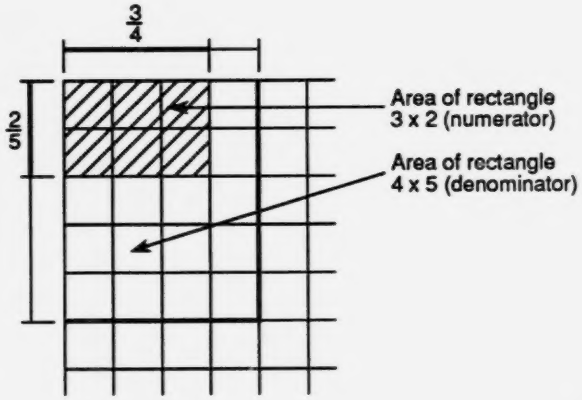


FIGURE 13.12
Relating the algorithm for multiplying fractions to the model.

There are several methods for developing understanding of division of fractions. Perhaps the most meaningful method is to use *compound* fractions.

EXAMPLE: $\frac{3}{4} \div \frac{1}{3} =$

STEP 1: Write the problem as a compound fraction where the dividend becomes the numerator and the divisor becomes the denominator.

$$\frac{\frac{3}{4}}{\frac{1}{3}}$$

STEP 2: Simplify the compound fraction. There are several methods for doing this, two of which are demonstrated here. Method A involves reciprocals and Method B involves the least common denominator.

Method A

STEP 3: Multiply numerator and denominator of the compound fraction by the reciprocal of the denominator (making the denominator of the compound fraction one).

$$\frac{\frac{3}{4} \times \frac{3}{1}}{\frac{1}{3} \times \frac{3}{1}} = \frac{\frac{3}{4} \times \frac{3}{1}}{1}$$

STEP 4: Simplify. $\frac{3}{4} \times \frac{3}{1} = \frac{9}{4}$

Method B

STEP 3: Multiply numerator and denominator of compound fraction by the least common denominator.

$$\frac{\frac{3}{4} \times 12}{\frac{1}{3} \times 12} = \frac{\frac{3}{4} \times \frac{12}{1}}{\frac{1}{3} \times \frac{12}{1}} = \frac{3 \times 3}{1 \times 4}$$

STEP 4: Simplify. $\frac{3 \times 3}{1 \times 4} = \frac{9}{4}$

■ A second method is an outgrowth of a common error made by students—they divide numerator by numerator and denominator by denominator. Nothing is wrong with this procedure if you know how to simplify the answer. This technique, however, is not very meaningful.

EXAMPLE: $\frac{5}{6} \div \frac{3}{2}$

STEP 1: Divide numerator by numerator and denominator by denominator.

$$\frac{5 \div 3}{6 \div 2}$$

STEP 2: Use one of the methods presented above to simplify.

$$\frac{\frac{5}{3}}{\frac{6}{2}} = \frac{\frac{5}{3} \times 6}{\frac{6}{2} \times 6} = \frac{10}{18}$$

This method for dividing fractions becomes very simple when the numerator and denominator in the first fraction are divisible by the numerator and denominator of the second, respectively.

EXAMPLE: $\frac{4}{6} \div \frac{2}{3}$

$$\frac{4 \div 2}{6 \div 3} = \frac{2}{2} = 1$$

When they are not, simply re-express the first fraction so that they are divisible.

EXAMPLE: $\frac{5}{6} \div \frac{3}{2}$

STEP 1: Re-express $^5/_6$ as an equivalent fraction where the numerator is divisible by 3 and the denominator is divisible by 2.

$$\frac{15}{18} \div \frac{3}{2}$$

STEP 2: Divide.

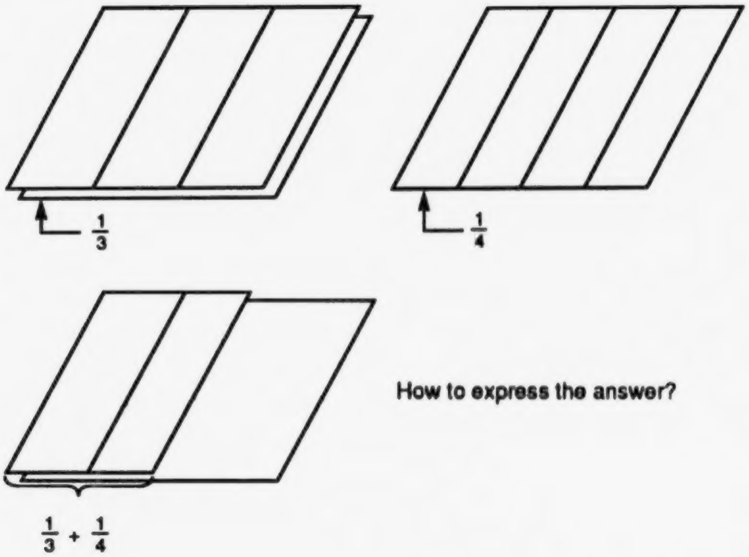
$$\frac{15 \div 3}{18 \div 2} = \frac{5}{9}$$

It is not necessary to teach all of these techniques to every student, but it is advisable that students understand more than one—when a student sees a problem from a variety of angles, chances of understanding are greatly improved.

Adding and Subtracting Fractions with Unlike Denominators (Objectives G, H)

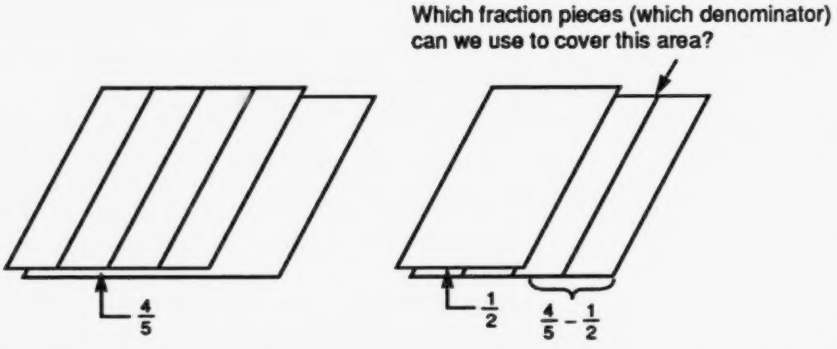
The first step in teaching students to add and subtract fractions with unlike denominators is to attempt to lead them to discover the need for common denominators. Ask a student to add one-third and one-fourth with the materials and when he has done this, ask him what the answer is. The student should discover that neither thirds nor fourths can be used to name the answer. (Next, have him experiment with the different fractional parts to find one that can be used to express the answer.) Thirds will not work because one-third plus one-fourth is less than two-thirds but greater than one-third (have the student use the fractional parts to check this out, as shown in Figure 13.13). Fourths will not work. Eventually, the student will discover that twelfths can be used to express the answer. Begin with easy problems like $^1/_2 + {}^1/_4$. Progress to problems like $^3/_5 + {}^1/_2$. Later include sums whose answer is greater than one (be sure to have appropriate materials available).

FIGURE 13.13
Discovering common denominators.


$\frac{1}{3}$
$\frac{1}{4}$
How to express the answer?
$\frac{1}{3} + \frac{1}{4}$

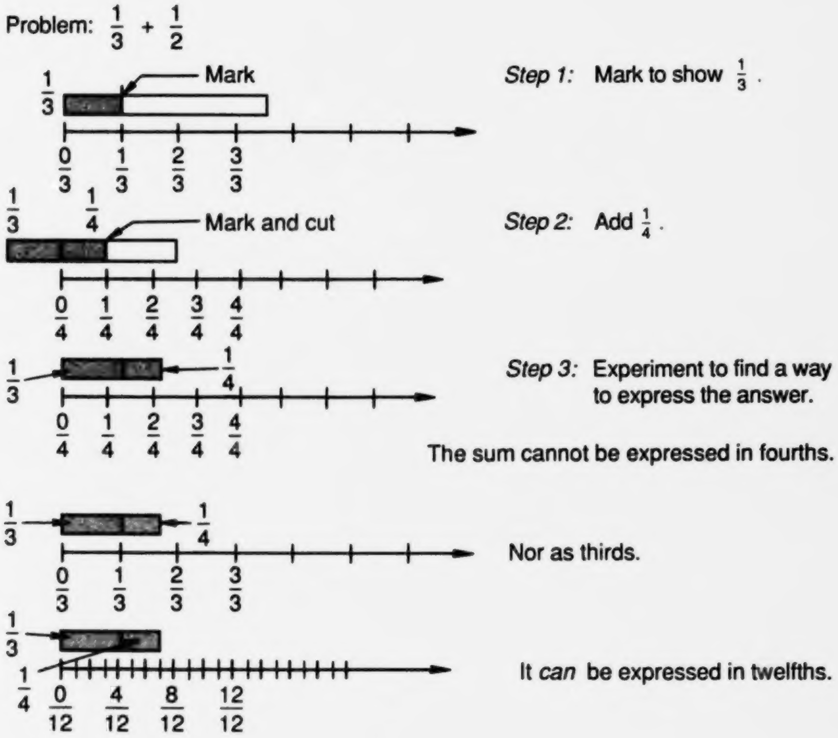
■ Have students do similar activities for subtraction. In a problem like $^4/_5 - {}^1/_2$, represent four-fifths by placing four of the fifths units side by side. Place a one-half unit over the four-fifths with either the left or right sides lined up (see Figure 13.14). The portion of the fifths that is not covered is the answer. Try to find a way to express the answer (the answer is $^3/_{10}$, since three-tenths cover up the portion of the four-fifths not covered by the one-half).

FIGURE 13.14
Subtracting fractions with unlike denominators.


Which fraction pieces (which denominator) can we use to cover this area?
$\frac{4}{5}$
$\frac{1}{2}$
$\frac{4}{5} - \frac{1}{2}$

■ The fraction table introduced in the last chapter can be extremely useful in developing understanding of addition and subtraction of fractions with unlike denominators. The steps are similar to those discussed above for fractions with like denominators. First, mark the strip of paper to indicate the first fraction. Place the mark on zero, cut the strip at the second fraction and the strip will physically represent the answer. Experiment with the different number lines until you find a way of expressing the answer. Figure 13.15 illustrates the use of the fraction table to find $^1/_3 + {}^1/_2$.

FIGURE 13.15
Adding unlike fractions using fraction strips.

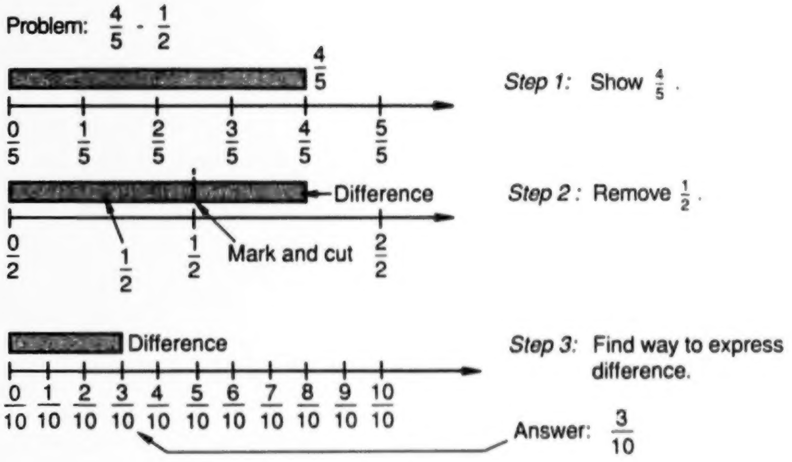

Problem: $\frac{1}{3} + \frac{1}{2}$
Mark
Step 1: Mark to show $\frac{1}{3}$.
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$
$\frac{1}{3}$ $\frac{1}{4}$
Mark and cut
Step 2: Add $\frac{1}{4}$.
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$
Step 3: Experiment to find a way to express the answer.
The sum cannot be expressed in fourths.
Nor as thirds.
It *can* be expressed in twelfths.
$\frac{0}{12}$ $\frac{4}{12}$ $\frac{8}{12}$ $\frac{12}{12}$

Of course, when teaching concepts like this with materials, each step should be presented and discussed with a number of different problems.

Do similar activities for subtraction of fractions. As in Figure 13.16, mark or cut the strip of paper to indicate the minuend. Remove the amount indicated by the subtrahend. Experiment to find a way to express the answer.

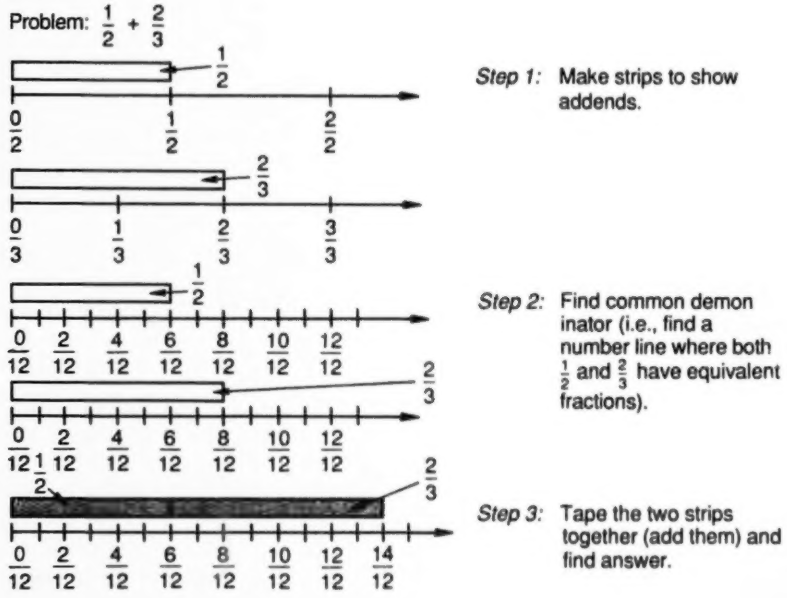
When we add or subtract fractions with materials, the fact that they have unlike denominators is of little concern: experimenting to find an appropriate denominator takes little more time than doing the entire problem on one number line (when the two fractions have the same denominator). At the symbolic level we must convert the fractions to those with common denominators *before* we add or subtract. Note that, in order to help children understand the addition and subtraction algorithms (with fractions), we could use techniques paralleling the algorithm on the fraction table.

FIGURE 13.16
Subtracting fractions on the fraction table.


Problem: $\frac{4}{5} - \frac{1}{2}$
$\frac{4}{5}$
$\frac{0}{5}$ $\frac{1}{5}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{4}{5}$ $\frac{5}{5}$
Step 1: Show $\frac{4}{5}$.
Difference
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{2}{2}$
Mark and cut
Step 2: Remove $\frac{1}{2}$.
Difference
$\frac{0}{10}$ $\frac{1}{10}$ $\frac{2}{10}$ $\frac{3}{10}$ $\frac{4}{10}$ $\frac{5}{10}$ $\frac{6}{10}$ $\frac{7}{10}$ $\frac{8}{10}$ $\frac{9}{10}$ $\frac{10}{10}$
Step 3: Find way to express difference.
Answer: $\frac{3}{10}$

- First, have them find fractions that are equivalent to those in the problem and that, at the same time, have common denominators, as illustrated in Figure 13.17. Next have them add or subtract the two fractions and express the answer.

FIGURE 13.17
Developing an understanding of addition and subtraction algorithms involving fractions.


Problem: $\frac{1}{2} + \frac{2}{3}$
$\frac{1}{2}$
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$
Step 1: Make strips to show addends.
$\frac{2}{3}$
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$
$\frac{1}{2}$
$\frac{0}{12}$ $\frac{2}{12}$ $\frac{4}{12}$ $\frac{6}{12}$ $\frac{8}{12}$ $\frac{10}{12}$ $\frac{12}{12}$
$\frac{2}{3}$
Step 2: Find common demon inator (i.e., find a number line where both $\frac{1}{2}$ and $\frac{2}{3}$ have equivalent fractions).
$\frac{0}{12}$ $\frac{2}{12}$ $\frac{4}{12}$ $\frac{6}{12}$ $\frac{8}{12}$ $\frac{10}{12}$ $\frac{12}{12}$
$\frac{1}{2}$
$\frac{2}{3}$
$\frac{0}{12}$ $\frac{2}{12}$ $\frac{4}{12}$ $\frac{6}{12}$ $\frac{8}{12}$ $\frac{10}{12}$ $\frac{12}{12}$ $\frac{14}{12}$
Step 3: Tape the two strips together (add them) and find answer.

The process of finding least common denominators at the symbolic level is often very difficult for students. One technique that most students find quite meaningful is to list the sequence multiples of each denominator until you find the multiple that is common to both strings of multiples. The *least common denominator* (LCD) of two or more fractions is the smallest number by which all the denominators are divisible. Any two fractions have an infinite number of common denominators (one of these is the product of the two denominators) but the smallest or *least* of these is the most useful in computing the sum.

EXAMPLE: $\frac{3}{4} + \frac{5}{6}$

STEP 1: Multiply each denominator by the sequence 1, 2, 3, 4, ...

4: 4, 8, 12, 16, 20, 24 . . .
6: 6, 12, 18, 24, 30, 36 . . .

STEP 2: Select the smallest or *least* of the common denominators (12).

STEP 3: Convert $^3/_4$ and $^5/_6$ to equivalent fractions whose denominators are each 12. Chapter 12 includes a detailed explanation of finding equivalent fractions.

STEP 4: Add.

Another technique is to have students multiply each fraction by the sequence $^1/_1$, $^2/_2$, $^3/_3$, $^4/_4$, etc., until they have found two fractions (equivalent to the original fractions) that have common denominators. This technique not only helps locate the LCD, but at the same time gives them the fractions that will added.

EXAMPLE: $\frac{5}{6} - \frac{3}{8}$

STEP 1: Multiply each fraction by the sequence $^1/_2$, $^2/_2$, $^3/_3$,. . . .

$$\frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{\textcircled{24}} = \frac{25}{30} = \ldots$$

$$\frac{3}{8} = \frac{6}{18} = \frac{9}{\textcircled{24}} = \ldots$$

until you have found the least common denominator.

STEP 2: Subtract.

The standard procedure for adding and subtracting fractions with unlike denominators involves the concept of equivalent fractions. Students need to understand that any fraction multiplied by one $(^n/_n)$ is equal to that fraction. They also need to be able to find the least common denominator of two fractions. Assuming students have understood the activities discussed in this chapter and in Chapter 12, it is possible to develop understanding of the standard procedure for adding and subtracting fractions. When we present these concepts for the first few times, it is a good idea to accompany them with manipulative or pictorial experiences. Figure 13.18 illustrates the procedure in the following problem.

PROBLEM: $\frac{5}{6} + \frac{3}{4} =$

STEP 1: Find the least common denominator (12).

STEP 2: Convert the fractions to their equivalents with the least common denominator and add. Point out the meaning of each step.

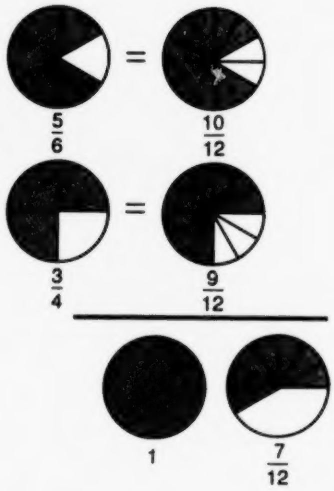
FIGURE 13.18
Use illustrations to demonstrate the meaning of adding fractions.

A. $\frac{5}{6} \times 1 + \frac{3}{4} \times 1$

B. $\frac{5}{6} \times \boxed{\frac{2}{2}} + \frac{3}{4} \times \boxed{\frac{3}{3}}$

C. $\frac{5}{6} \times \boxed{\frac{2}{2}} + \frac{3}{4} \times \boxed{\frac{3}{3}}$

D. $\frac{10}{12} + \frac{9}{12} = \frac{19}{12}$

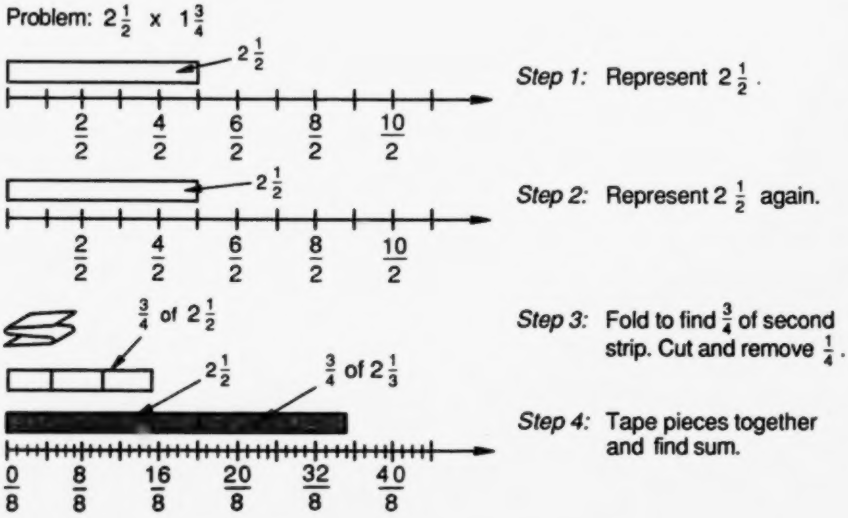

=
5/6
10/12
=
3/4
9/12
1
7/12

Mixed and Improper Fractions (Objectives G-J)

Operations with mixed fractions should be taught along with proper fractions. In most cases, the manipulations discussed above can be extended easily to include mixed numbers and improper fractions.

An extended fraction table is useful for finding products of mixed numbers. A product like $2^1/_2 \times 1^3/_4$ would be solved similarly to a problem involving two proper fractions—cut a strip to represent $2^1/_2$, then fold and cut three-fourths of it and use the number lines to find a way to express the answer. To find the product of $2^1/_2$ and $1^3/_4$ (see Figure 13.19), cut a strip to represent $2^1/_2$. Since we want one plus three-fourths of this strip, place the first $2^1/_2$ strip aside to cut another strip $2^1/_2$ units long; then find three-fourths of the second strip using the usual techniques. Tape the two together and experiment to find the answer.

FIGURE 13.19
A fraction table for mixed numbers.

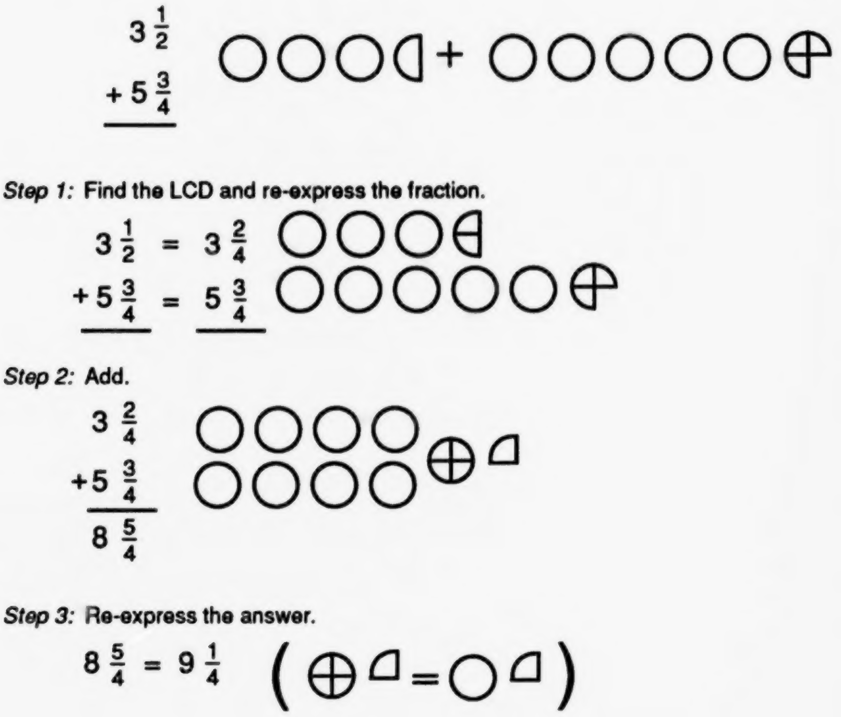

Problem: 2 1/2 x 1 3/4
2 1/2
Step 1: Represent 2 1/2.
2/2
4/2
6/2
8/2
10/2
2 1/2
Step 2: Represent 2 1/2 again.
2/2
4/2
6/2
8/2
10/2
3/4 of 2 1/2
Step 3: Fold to find 3/4 of second strip. Cut and remove 1/4.
2 1/2
3/4 of 2 1/3
Step 4: Tape pieces together and find sum.
0/8
8/8
16/8
20/8
32/8
40/8

Dividing mixed and improper fractions on the fraction table would be done no differently from dividing proper fractions.

Present addition and subtraction of fractions in vertical form. Associate these problems with pictures or manipulative materials.

EXAMPLE: $\begin{array}{r} 3\tfrac{1}{2} \\ +\ 5\tfrac{3}{4} \\ \hline \end{array}$ (Figure 13.20)

FIGURE 13.20
Adding mixed numbers.


$3\frac{1}{2}$
$+5\frac{3}{4}$
+
Step 1: Find the LCD and re-express the fraction.
$3\frac{1}{2} = 3\frac{2}{4}$
$+5\frac{3}{4} = 5\frac{3}{4}$
Step 2: Add.
$3\frac{2}{4}$
$+5\frac{3}{4}$
$8\frac{5}{4}$
Step 3: Re-express the answer.
$8\frac{5}{4} = 9\frac{1}{4}$
(=)

When dealing with operations (except division) on mixed fractions like the example above, we can use many of the principles that helped develop meaning for operations on whole numbers. These principles are expanded notation and the commutative, associative, and distributive properties.

EXAMPLE: $3\frac{1}{2} + 5\frac{3}{4}$

SOLUTION:

$(3 + \frac{1}{2}) + (5 + \frac{3}{4})$

$(3 + 5) + (\frac{1}{2} + \frac{3}{4})$

$8 + \frac{5}{4}$

$8 + (1 + \frac{1}{4})$

$(8 + 1) + \frac{1}{4}$

$9 + \frac{1}{4} = 9\frac{1}{4}$

EXAMPLE: $5\frac{2}{3} \times 2\frac{1}{8}$

SOLUTION:

$$(5 + \tfrac{2}{3}) \times (2 + \tfrac{1}{8})$$

$$5 \times (2 + \tfrac{1}{8}) + \tfrac{2}{3} \times (2 + \tfrac{1}{8})$$

$$10 + \tfrac{5}{8} + \tfrac{4}{3} + \tfrac{2}{24}$$

$$10 + \tfrac{49}{24}$$

$$10 + (2 + \tfrac{1}{24}) = 12\tfrac{1}{24}$$

Similar procedures can be done vertically.

EXAMPLE:

$$\begin{array}{r} 5\frac{2}{3} \\ \underline{2\frac{1}{8}} \end{array}$$

SOLUTION: (*List all partial products.*)

Estimating Answers to Operations with Fractions (Objective L)

Perhaps the best method for helping children estimate answers to problems involving fractions is to have them visualize the operations on number lines. For example, to estimate the sum of $\frac{1}{3}$ and $\frac{3}{4}$ picture $\frac{3}{4}$ on the number line, then imagine going $\frac{1}{3}$ of a unit beyond that point. The answer would be near $1\frac{1}{4}$. Students may beome even better estimators by visualizing numbers and "rounding" the fractions involved to those with like denominators. For example, $\frac{1}{3} + \frac{3}{4}$ may be estimated by rounding $\frac{1}{3}$ to either $\frac{1}{4}$ or $\frac{2}{4}$, then adding the numerators. When estimating sums or differences of larger mixed numbers, simply round by deletion to the nearest whole, ten, hundred, etc. and estimate the sum or difference of the whole numbers.

To estimate products of fractions, have students visualize operations on fraction strips and/or round the fractions to those that may be operated upon mentally. For example, the product, $\frac{1}{3} \times \frac{7}{8}$ can be rounded to $\frac{1}{3} \times 1$ and solved mentally. To solve division problems involving fractions, estimate the number of times the second fraction may be subtracted from the first. Sometimes, it may be useful to round the fractions first.

■ Give students a variety of problems to solve where they must estimate sums, differences, products, and quotients of fractions. Include verbal problems. After they have estimated the answers, have them determine how accurate their estimates were.

■ Appendix F of this book contains a BASIC program called "ESTFRAC." The purpose of this program is to provide practice in estimating operations on fractions. The program prints a fraction problem, such as $\frac{3}{4} + \frac{1}{3}$, on the screen. The student must estimate the sum and type the answer into the computer. If the fraction input is within 25 percent in either direction of the correct answer, the student is rewarded. Otherwise, he is asked to enter another answer. It is suggested that you have manipulative materials like the fraction strips for Fraction Bars® available so that students will be able to use them when they have difficulty estimating a sum.

Using Calculators and Computers to Operate on Fractions (Objectives M, N)

Fractions may be operated upon with calculators and computers by using the divide key or operation to convert the fractions to decimals. It is important for childen to understand the meaning of decimal numerals (see the previous chapter) so that they will understand the intermediate and final results displayed. The sequence of keystrokes and displays for $12\frac{7}{8} + 7\frac{1}{6}$ are:

12 **M+** 7 ÷ 8 = **M+** 7 **M+** 1 ÷ 6 = **M+ MR**

As was mentioned in Chapter 3, a fundamental understanding that will allow the student to use the calculator to do this operation is that $12\frac{7}{8} = 12 + \frac{7}{8}$. The problem may be simplified even further if the child realizes that $12 + \frac{7}{8} = \frac{7}{8} + 12$ (the commutative property for rational numbers):

7 ÷ 8 + 12 = **M+** 1 ÷ 6 + 7 = **M+ MR**

Similar sequences may be used when subtracting, multiplying, or dividing fractions. Remember, however, the lack of commutativity for subtraction and division.

3 ÷ 8 + 5 **M+** 6.5 − **MR** =

To solve operations on fractions on the computer, you must remember that many computers will interpret a number like 3 1/2 as 31 divided by 2. The Apple computer, for example, ignores spaces between digits in BASIC programs. Others will give you an error message if you enter the statement PRINT 3 1/2 + 5 7/8. In order to have the computer evaluate expressions involving fractions correctly, write mixed numbers as sums. To get the computer to solve the expression 3 1/2 + 5 7/8, enter PRINT 3 + 1/2 + 5 + 7/8.

■ Give children a variety of exercises and verbal problems involving fractions to solve using calculators and PRINT statements in Logo or BASIC.

The most important aspect of developing proficiency with fractions is to develop understanding of what fractions are and what the operations on them mean. When developing proficiency for students with operations on fractions, do not forget calculators and computers. Indeed, in this age of technology, such skills are at least as important as paper and pencil skills.

SAMPLE DRILL AND PRACTICE ACTIVITIES

Activity 13.1 Fractions Math-O

Objective: To add fractions with unlike denominators.

Materials: Math-O game board (Figure 20.6) and materials
Math-O cards with problems involving addition of fractions

Methods: See the instructions on the game board in Figure 20.6.

Variations: Math-O cards for any operation (addition, subtraction, multiplication and division) on fractions can be used separately or together.

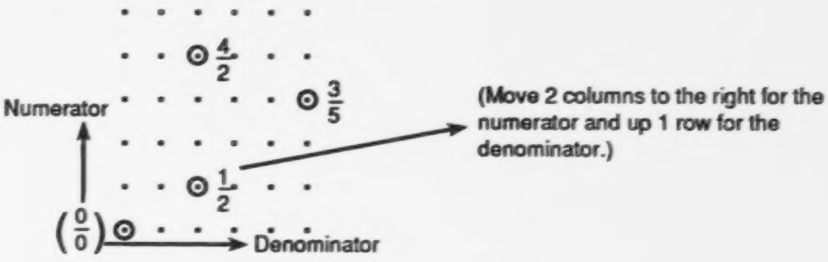
Activity 13.2 Fractions Connect-the-Dots

Objective: To divide proper fractions.

Materials: Puzzle sheet (Figure 13.22)

Methods: Students work the problems and connect the dots in the sequence indicated by the order of the problems. On the dot paper, each geoboard dot represents a different fractions. Denominators are found by moving to the right and numerators by moving up on the dot paper. The dot in the lower left corner of the page is the (nonexistent) fraction $^{0}/_{0}$. Several fractions are illustrated on the sample geoboard in Figure 13.21.

FIGURE 13.21
Showing fractions on a geoboard.



(Move 2 columns to the right for the numerator and up 1 row for the denominator.)

SUGGESTED READINGS

Arithmetic Teacher 31 (6), February 1984. This issue is dedicated to rational numbers and has several articles about operations on fractions.

Carlisle, E. Fractions with popsicle sticks. *Arithmetic Teacher* 27 (6), February 1980: 50–51.

Moulton, J. P. A working model for rational numbers. *Arithmetic Teacher* 22 (4), April 1975: 328–332.

Sweetland, R. D. Understanding multiplication of fractions. *Arithmetic Teacher* 32 (1), September 1984: 48–52.

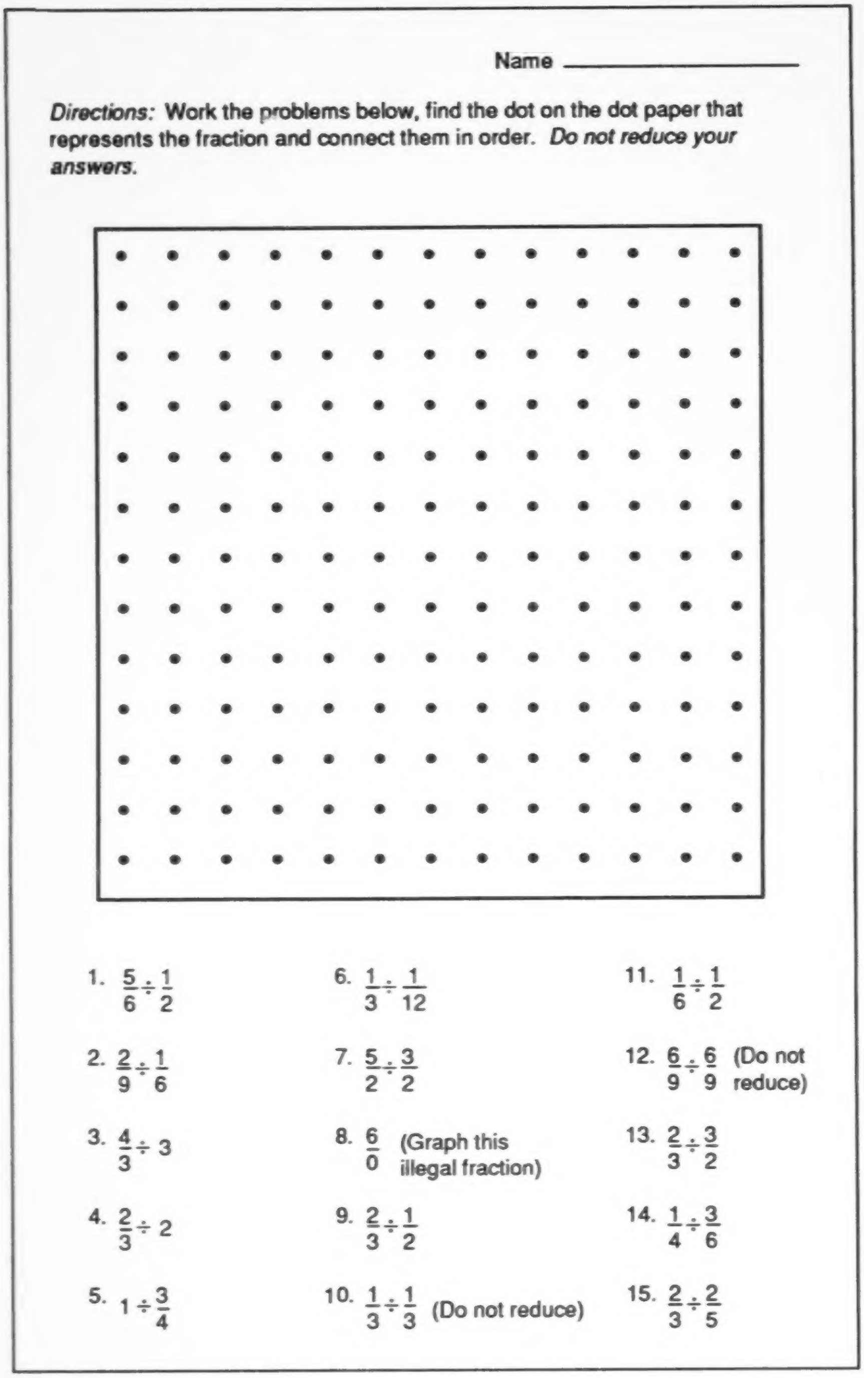
Thompson, C. Teaching division of fractions with understanding. *Arithmetic Teacher* 27(5), January 1979: 24–27.

Van de Walle, J., and Thompson, C. S. Let's do it: Fractions with fraction strips. *Arithmetic Teacher* 32 (4), December 1984: 49.

________. Fractions with counters. *Arithmetic Teacher* 28 (2), October 1980: 6–11.

Wiebe, J. H. Discovering fractions on a "fraction table." *Arithmetic Teacher* 33 (4), December 1985: 49–51.

FIGURE 13.22
A connect-the-dot worksheet for fractions.



Name ______________________

Directions: Work the problems below, find the dot on the dot paper that represents the fraction and connect them in order. *Do not reduce your answers.*

1. $\frac{5}{6} \div \frac{1}{2}$
2. $\frac{2}{9} \div \frac{1}{6}$
3. $\frac{4}{3} \div 3$
4. $\frac{2}{3} \div 2$
5. $1 \div \frac{3}{4}$
6. $\frac{1}{3} \div \frac{1}{12}$
7. $\frac{5}{2} \div \frac{3}{2}$
8. $\frac{6}{0}$ (Graph this illegal fraction)
9. $\frac{2}{3} \div \frac{1}{2}$
10. $\frac{1}{3} \div \frac{1}{3}$ (Do not reduce)
11. $\frac{1}{6} \div \frac{1}{2}$
12. $\frac{6}{9} \div \frac{6}{9}$ (Do not reduce)
13. $\frac{2}{3} \div \frac{3}{2}$
14. $\frac{1}{4} \div \frac{3}{6}$
15. $\frac{2}{3} \div \frac{2}{5}$

QUESTIONS

1. Some claim that if it were not for our measuring system, we would not need to learn operations on fractions—in the rare nonmeasurement situations calling for operations on fractions, we could convert the fractions to decimals and then perform the operations. In light of this statement, discuss the place of operations on fractions in the curriculum.
2. Write a computer program that will accept the numerator and denominator of two fractions as four separate inputs, find the least common denominator of the two fractions, then find and print out their sum (expressed in lowest terms).
3. Solve the following on your calculator.
   a. $^7/_8 + {}^4/_5$
   b. $35^2/_7 \times 47^1/_6$
   c. $3^1/_2 + 5^3/_8 \times 14^2/_3$
4. Discuss the use of circles vs. rectangles to model fractions. Which is better? Why?
5. Develop a lesson about operations on fractions that makes use of food as a model. The food should not be messy so that it can be consumed after being manipulated.
6. Interview a number of people to find out when (if ever) and in what situations they have divided fractions outside of school. Discuss the results. Should we continue to teach the "invert and multiply" algorithm?

CHAPTER

14

Decimal Operations, Ratios, Proportions, & Percentages

With the metric system, calculators, and computers becoming a major part of our lives, the ability to perform decimal computations is ever more important. Percentages are no less important since we use them continuously in business, and in any instance when we wish to express fractions involving large numbers in a manner that is easily comprehensible.

Chapter 12 shows how the meaning of decimal and percentage numerals can be developed. These meanings are the foundation on which operations are built—make sure that students thoroughly understand the numerals they are working with before asking them to add, subtract, multiply, and divide with them.

Behavioral Objectives for Operations on Decimals and Percentages

A. Add decimals in tenths and hundredths with no digits to the left of the decimal.

B. Subtract decimals in tenths and hundredths with no digits to the left of the decimal.

C. Multiply decimals in tenths and hundredths with no digits to the left of the decimal.

D. Divide a decimal or a whole number by a decimal in tenths or hundredths—no annexation of zeroes.

E. Add two or more decimals, each containing any number of digits, including decimals greater than one.

F. Subtract a decimal from another (any number of digits).

G. Multiply two decimals (any number of digits).

H. Divide any decimal by any other decimal including those where it is necessary to annex zeroes.

I. Solve verbal problems involving decimals.

J. Estimate answers to operations on decimals.

K. Write numerals to express ratios and use concrete materials to represent written ratios.
L. Represent proportions with models.
M. Solve for missing values in proportions.
N. Represent percentage numerals with models.
O. Find the missing value in percentage problems.
P. Solve verbal problems involving percentages.
Q. Solve percentage problems where A is a mixed percentage and/or is greater than 100%.
R. Estimate answers to problems involving percentages.

PROPERTIES OF OPERATIONS ON RATIONAL NUMBERS

In Chapter 12, rational numbers were defined and in Chapters 10 and 11, the properties of addition, subtraction, multiplication, and division related to whole numbers were discussed. All these properties—commutativity, the zero property, and so on—are relevant to rational numbers as well as to whole numbers. The following summarizes these properties as they relate to rational numbers:

(1) Commutative Property:

Addition: $.25 + 3.06 = 3.06 + .25$

Multiplication: $.333 \times 90 = 90 \times .333$

Note: Subtraction and division of rational numbers are not commutative.

(2) Associative Property:

Addition: $(.405 + 1.2) + 235.8 = .405 + (1.2 + 235.8)$

Multiplication: $(.4 \times 16) \times .5 = .4 \times (16 \times .5)$

Note: Subtraction and division of rational numbers are not associative.

(3) Distributive Property:

Multiplication over addition:

$.5 \times (25 + .45) = (.5 \times 25) + (.5 \times .45)$

Division over addition (only from right side):

$(12 + .75) \div .2 = (12 \div .2) + (.75 \div .2)$

(4) Zero Property:

$2.45 \times 0 = 0 \times 2.45 = 0$

(5) Identity Property:

Addition	$.45 + 0 = 0 + .45 = .45$
Subtraction	$.333333\ldots - 0 = .33333\ldots$ (right only)
Multiplication	$25.5 \times 1 = 1 \times 25.5 = 25.5$
Division	$.8 \div 1 = .8$ (right side only)

Note: The above examples could be given using fractions instead of decimals since any fraction may be expressed as a decimal and vice versa.

Review these properties and discuss their generalization to rational numbers. They are used to develop meaning for a variety of operations.

TEACHING OPERATIONS ON DECIMALS

Manipulative Materials for Decimal Operations (Objectives A–D)

Manipulative and pictorial activities for teaching operations with decimals are, for the most part, an extension of the activities for developing understanding of decimal numerals and operations on whole numbers. If children understand what decimals are and if they understand the meaning of the operations, most children should be able to move quickly from manipulative and pictorial activities to symbolic activities for decimals.

The best manipulative for teaching addition and subtraction of decimals is base-ten blocks. To model operations on decimals, after representing the decimals (remember, as shown in Figure 14.1, the large cubes are the units, the "flats" represent the tenths, etc.), you manipulate the blocks the same way you do to model addition or subtraction of whole numbers. To model addition of decimals, represent each addend, join the sets, regroup if necessary, and then write the answer.

Another model (see Figure 14.2) for addition of fractions is the number line. Use a number line(s) marked in tenths and/or hundredths. Mark a strip of paper to show the first addend, place the mark on zero and mark the second addend, and so on for each addend in the problem, and cut the strip to show the sum. Now, place the left end of the strip on zero and read the sum on the number line at the right end.

To subtract one decimal from another, use the set separation meaning of subtraction as in Figure 14.3. First, represent the minuend, then remove the amounts represented by the subtrahend (after regrouping, if necessary). The remaining value is the difference.

To subtract decimals using the number line, place the left end of the strip on zero and mark and cut the strip to represent the minuend, as shown in Figure 14.4. Now, place the left end of the strip (representing the minuend) at zero and mark to represent the subtrahend. Cut at that mark and discard the piece representing the subtrahend. The remaining portion is the difference.

■ Give students a variety of problems to solve using these models. Be sure to give problems that are within the limits of the materials.

■ Base-ten blocks are of limited value for modeling multiplication and division of decimals, except when one factor or the divisor is a small whole number as in the

FIGURE 14.1
Adding decimals with base-ten blocks.

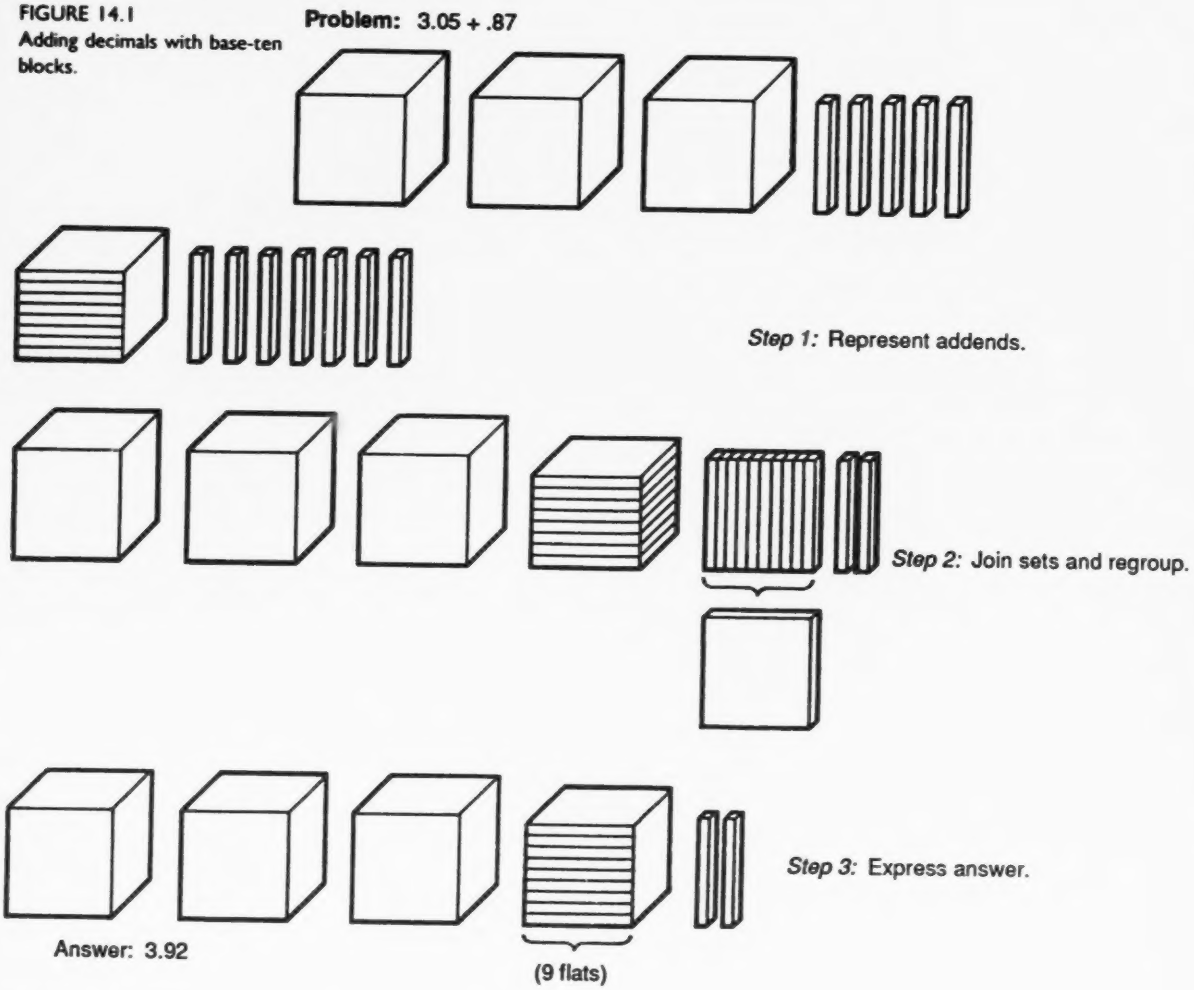

Problem: 3.05 + .87
Step 1: Represent addends.
Step 2: Join sets and regroup.
Step 3: Express answer.
Answer: 3.92
(9 flats)

FIGURE 14.2
Adding fractions on number lines.

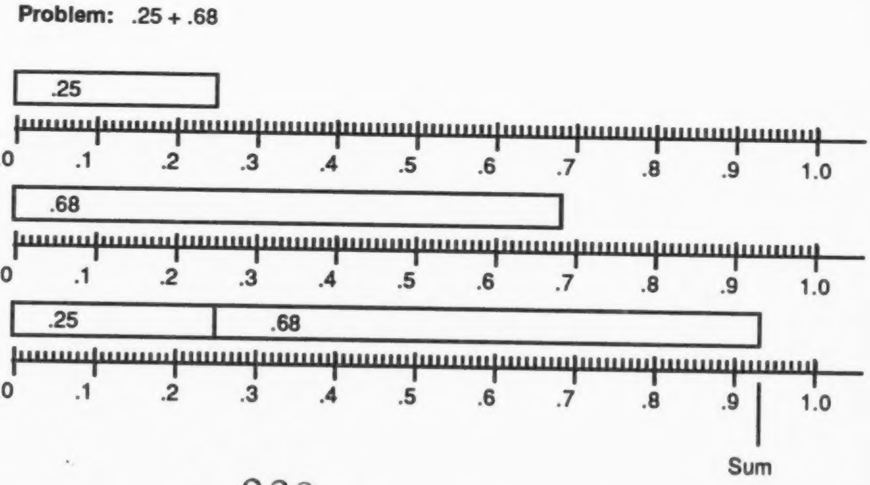
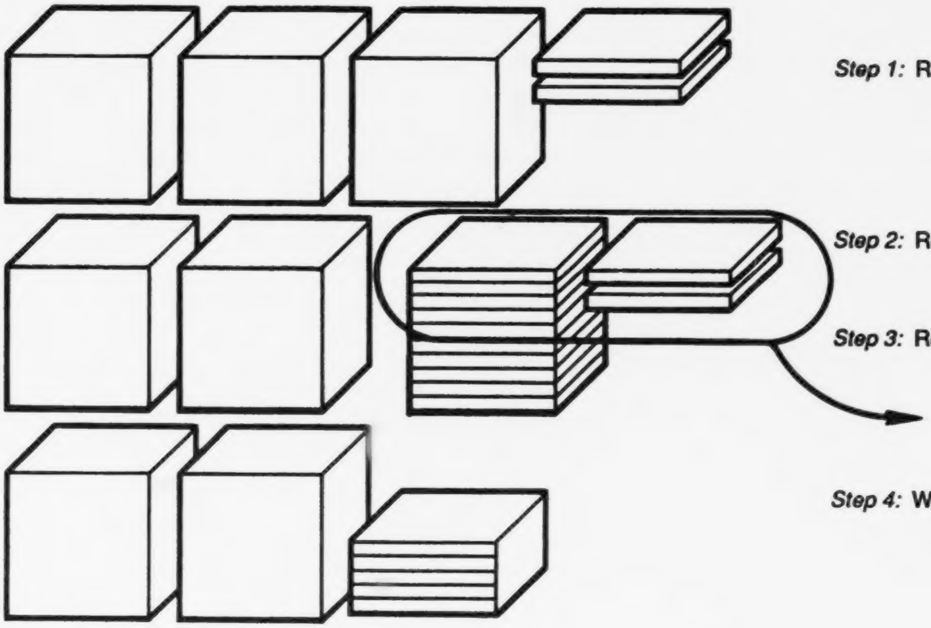

Problem: .25 + .68
.25
.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
.68
.25
.68
Sum

FIGURE 14.3
Subtracting decimals with base-ten blocks.

Problem: 3.2 − .7



Step 1: Represent minuend.

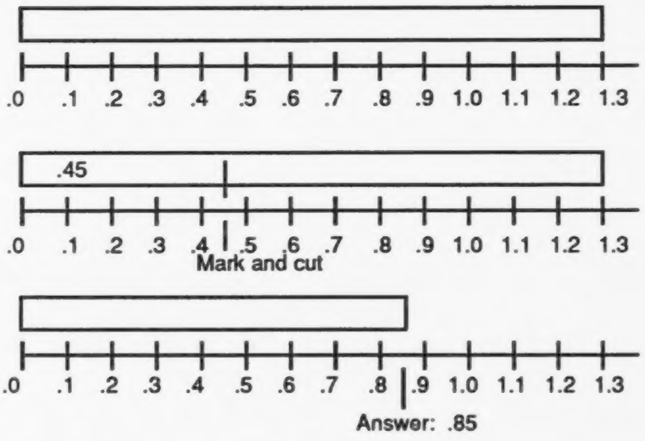
Step 2: Regroup.

Step 3: Remove subtrahend.

Step 4: Write difference.

FIGURE 14.4
Subtracting decimals with the number line.

Problem: 1.3 − .45


.0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3
.45
.0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3
Mark and cut
.0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3
Answer: .85

Step 1: Represent minuend.

Step 2: Remove subtrahend

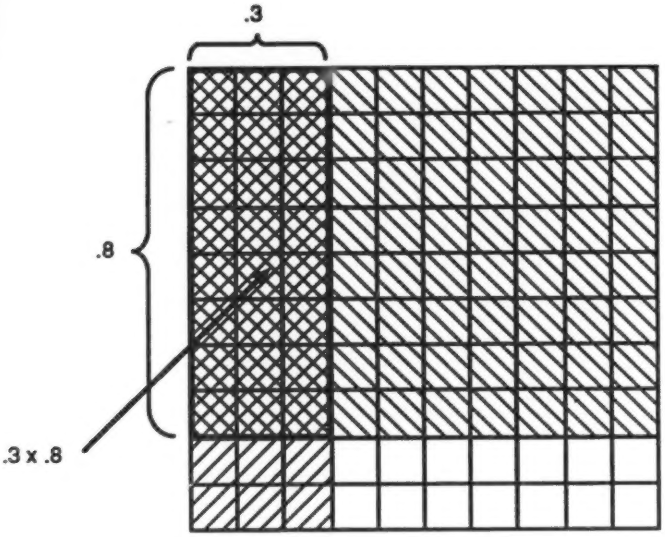
Step 3: Find answer.

problems 1.25 × 3 and .75 ÷ 5. For the former, you would represent 1.25 three times, join the sets, regroup, and express the answer. For the latter, you would partition .75 into 5 equal sets.

■ To model tenths times tenths (e.g., .3 × .8 as illustrated in Figure 14.5, you may use centimeter grid paper (marked in light blue). First, mark a 10 by 10 square to show a whole (or *unit*). Next, mark 1 by 10 vertical rectangles to show the first factor and shade the area with crayons or some other type of marker. Discuss how this represents the first factor. Next, find eighth-tenths of that area (or whatever the second factor of the product is) by marking 1 by 10 rectangles in the other direction. The product is the portion of the shaded area within the new area you have just found.

FIGURE 14.5
Using grids to mutiply decimals.

Problem: .3 x .8



Answer: 24 hundredths = .24

■ Number lines and strips of paper may also be used to develop meaning for multiplying decimals. Fig. 14.6 shows the sequence of manipulations.

■ To model division of decimals, use the idea of repeated subtraction and number lines with strips of paper. For example, in the problem .75 ÷ .25, the question is, how many times can I subtract .25 from .75? (See Fig. 14.7.) If your divisor is larger than the dividend, the answer will be a decimal (see Figure 14.8).

Since the manipulations involving multiplication and division are somewhat difficult, you may wish to do them on an overhead projector, as in Figure 14.9, or using a large felt board, then discuss the manipulations as a group. Simply draw the required number lines on a noncolored transparency, then cut your strips from colored transparencies. Mark the strips with permanent or nonpermanent transparency markers and cut with ordinary scissors.

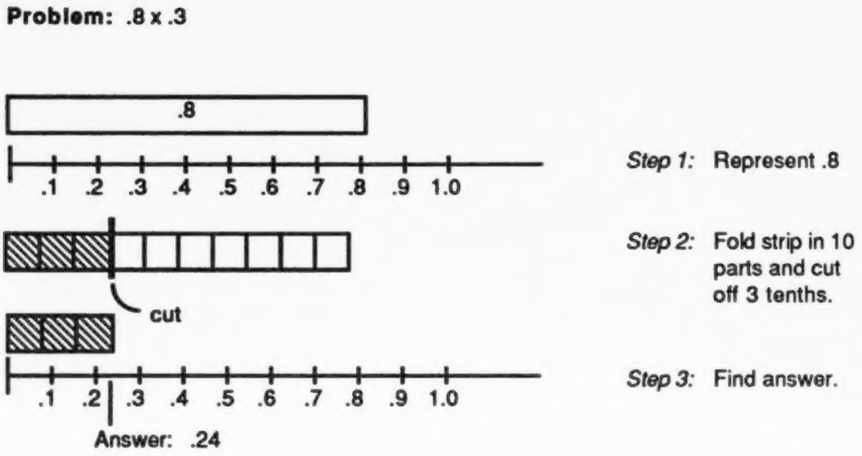


FIGURE 14.6
Number lines for multiplying fractions.

Problem: .8 x .3
.8
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Step 1: Represent .8
cut
Step 2: Fold strip in 10 parts and cut off 3 tenths.
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Answer: .24
Step 3: Find answer.

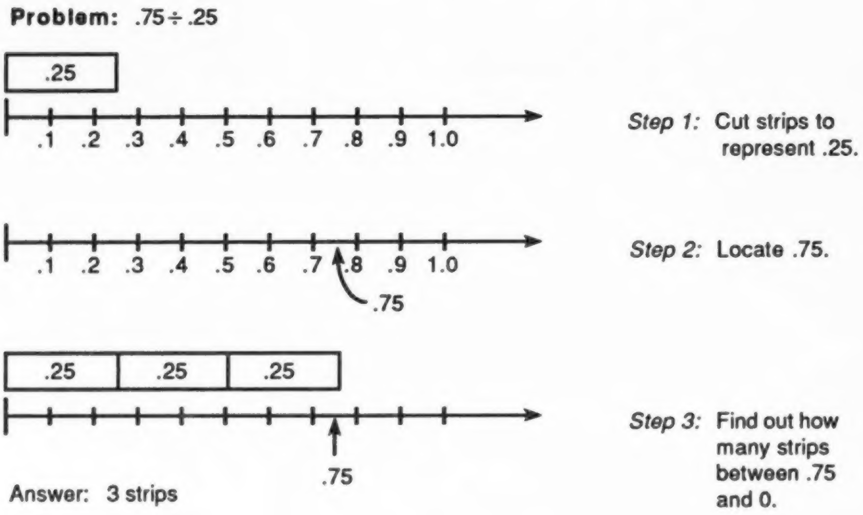


FIGURE 14.7
Number lines for .75 ÷ .25.

Problem: .75 ÷ .25
.25
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Step 1: Cut strips to represent .25.
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
.75
Step 2: Locate .75.
.25 .25 .25
.75
Answer: 3 strips
Step 3: Find out how many strips between .75 and 0.

Of course, as is discussed in the last chapter, it is not easy, or even necessarily desirable, to obtain exact answers to multiplication and division problems using number lines. Rather, their purpose is to develop understanding of the operations and give the students mental images that will help them estimate answers. Keep the limitations of all models in mind when using them and remember that the ultimate goal is understanding.

Symbolic Activities for Operations on Decimals (Objectives A–H)

One way to develop meaning for decimal operations is through fractions. In this method, decimals are converted to fractions, the operations are performed on the fractions, and the answers are converted back to decimals. This technique helps develop understanding of the major difficulty students face when working with de-

FIGURE 14.8
Number lines for .6 ÷ 1.2.

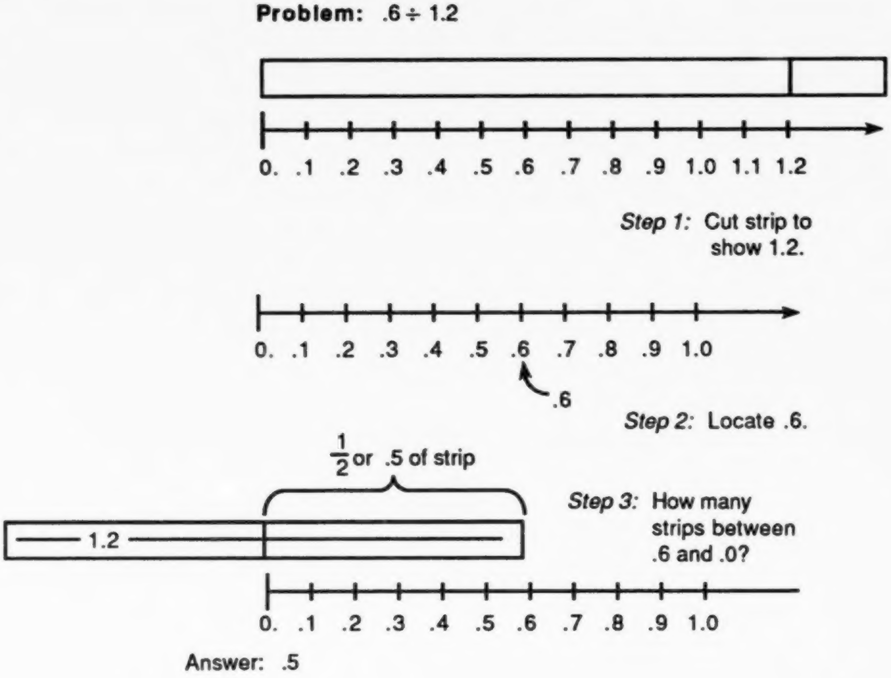
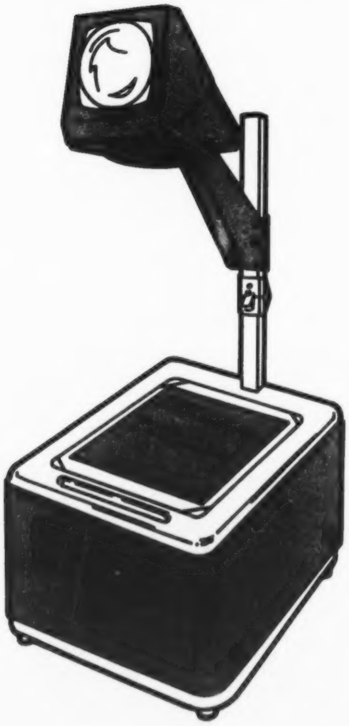

Problem: .6 ÷ 1.2
0. .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2
Step 1: Cut strip to show 1.2.
0. .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
.6
Step 2: Locate .6.
1/2 or .5 of strip
1.2
Step 3: How many strips between .6 and .0?
0. .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Answer: .5

FIGURE 14.9
Transparent number lines.



cimals—where to put the decimal in the answer. Here are examples for all four operations:

EXAMPLE 1: $.34 + .9 = ?$

$$\frac{34}{100} + \frac{9}{10} = \frac{34}{100} + \frac{90}{100} = \frac{124}{100} = 1.24$$

EXAMPLE 2: $1.3 - .29 =$

$$\frac{13}{10} - \frac{29}{100} = \frac{130}{100} - \frac{29}{100} = \frac{101}{100} = 1.01$$

EXAMPLE 3: $1.05 \times .6$

$$\frac{105}{100} \times \frac{6}{10} = \frac{630}{1000} = .630$$

EXAMPLE 4: $.25\overline{)12.5}$

METHOD 1: $$\frac{125}{10} \div \frac{25}{100} = \frac{125}{10} \times \frac{100}{25} = \frac{5}{1} \times \frac{10}{1} = 50$$

METHOD 2: $$\frac{12.5}{.25} \times \frac{100}{100} = \frac{1250}{25} \rightarrow 25\overline{)1250}$$

METHOD 3: (After the student has been introduced to Method 2)

$$.25\overline{)12.5} \rightarrow .25 \times 100\overline{)12.5 \times 100} \rightarrow 25\overline{)1250}$$

Another method for developing understanding of decimal operations is to use expanded notation and the commutative, associative, and distributive properties, as was done with operations on whole numbers. Below are examples of addition, subtraction, and multiplication of decimals.

EXAMPLE 1: $.34 + .9 = ?$

$(.3 + .04) + .9$

$(.3 + .9) + .04$

$1.2 + .04 = 1.24$

EXAMPLE 2: $1.3 - .29 = ?$

$$\begin{array}{r} 1 + .3 \\ \underline{-(\quad .2 + .09)} \\ \end{array} \rightarrow \begin{array}{r} \text{(Regroup)} \\ 1 + .2 + .10 \\ \underline{-(\quad .2 + .09)} \\ 1 + 0 + .01 = 1.01 \end{array}$$

EXAMPLE 3: $.24 \times 1.6 = ?$

METHOD 1:

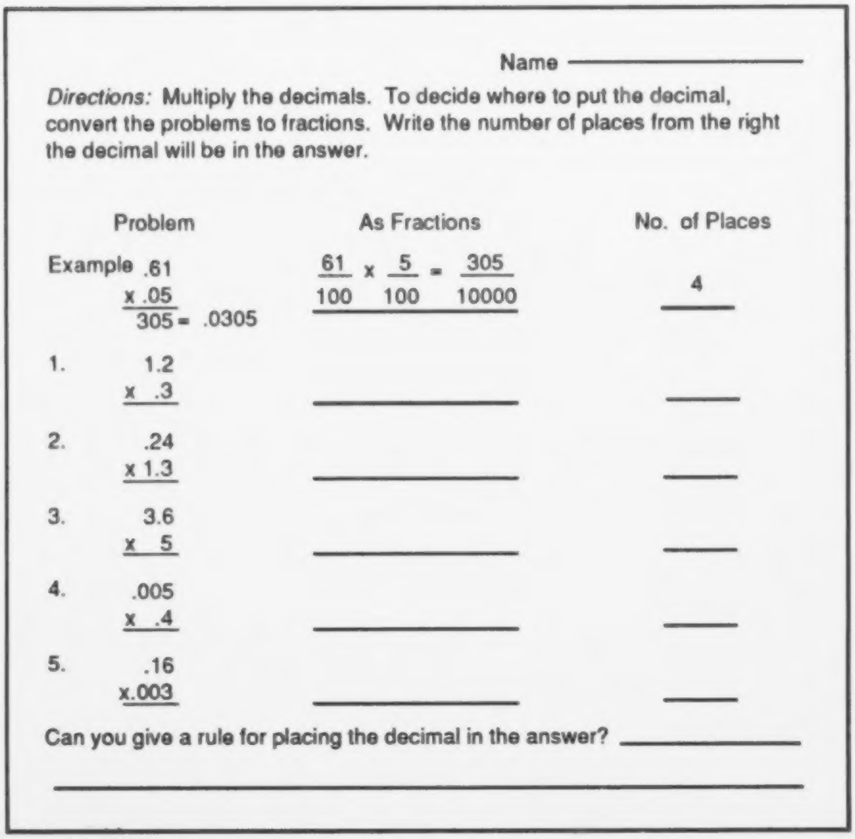
$$\begin{aligned} & (.2 + .04) \times (1 + .6) \\ &= .2 \times (1 + .6) + .04 \times (1 + .6) \\ &= .2 + .12 + .04 + .024 \\ &= .384 \end{aligned}$$

METHOD 2: (Listing partial products)

$$\begin{array}{r} .24 \\ \times\ 1.6 \\ \hline .024 \\ .12 \\ .04 \\ .2 \\ \hline .384 \end{array} \quad \text{Partial Products}$$

In problems like those in Example 3 just above, you may want to return to the use of fractions to help students understand and remember where to place the decimal in the answer. In fact you may lead them to discover the rule for placing the decimal through a set of structured problems like those in Figure 14.10.

FIGURE 14.10
Worksheet for discovering the decimal placement rule in the multiplication of decimals.



Name ____________

Directions: Multiply the decimals. To decide where to put the decimal, convert the problems to fractions. Write the number of places from the right the decimal will be in the answer.

	Problem	As Fractions	No. of Places
Example	$.61 \times .05 = 305 = .0305$	$\frac{61}{100} \times \frac{5}{100} = \frac{305}{10000}$	4
1.	$1.2 \times .3$	____________	____
2.	$.24 \times 1.3$	____________	____
3.	3.6×5	____________	____
4.	$.005 \times .4$	____________	____
5.	$.16 \times .003$	____________	____

Can you give a rule for placing the decimal in the answer? ____________

____

It is not necessary for each student to completely master more than one meaningful method for solving computational problems with decimals, but it may be a good idea for many students to be exposed to each method and to follow as you go through the steps. One of these techniques for each operation, however, should be selected and mastered before moving on to the standard algorithms.

Estimating Answers to Operations on Decimals (Objective J)

The mental algorithms for addition and subtraction of decimals are similar to those for whole numbers. Focus on the front digits of the values in the problem after checking their place values. For example, when estimating the sum of .5, .75, and .006, we would add .5 and .7 to get 1.2, ignoring the .006 since the place value of its first non-zero digit is less than that of the other two addends.

To estimate products and quotients of decimals, we can, as in making other estimates, focus on the front digits, but we must realize what happens to the place values of our answers. To help develop this understanding, we first focus on multiplying a variety of numbers by .1, then by .01, and so forth. Gradually, we want the students to realize that when we multiply by .1 we move the decimal one place to the left, when we multiply by .01 we move it 2 places, etc. When we divide by a .1 we move the decimal one place to the right, and so on. Here is the sequence for estimating quotients and products of decimals:

ESTIMATE: $.082 \times .5 =$

STEP 1: Multiply the first non-zero digits:

$$8 \times 5 = 40$$

STEP 2: Since you are multiplying hundredths by tenths, move the decimal three places to the left:

$$\left(\frac{1}{100} \times \frac{1}{10} = \frac{1}{1000} = .001 \right)$$

$$\underset{\smile\smile\smile}{.040}$$

ESTIMATE: $.726 \times .04$

STEP 1: $7 \times 4 = 28$

STEP 2: Tenths × hundredths = thousandths. (.726 = .7)

$\underset{\smile\smile\smile}{.028}$ (Move decimal three places to the left.)

ESTIMATE: $.08 \div .02$

STEP 1: $8 \div 2 = 4$

STEP 2: Hundredths ÷ hundredths = ones. (We do not need to move the decimal in the answer.)

$$\left(\frac{8}{100} \div \frac{2}{100} = \frac{8 \div 2}{100 \div 100} = \frac{4}{1} = 4 \right)$$

ESTIMATE: $1.25 \div .5$

STEP 1: $1 \div 5 = .2$

STEP 2: Ones ÷ tenths = tens.
Move decimal one place to *right*.
$\underset{\curvearrowright}{.2} = 2$

- When teaching students to estimate products and quotients of decimals, start with problems having factors and divisors of .1, .01, etc. (e.g., $.25 \times .1$, $2.45 \div .01$). Next, give them problems that can be rounded to these ($.45 \times .099$ or $1.2 \div .01045$). Next, move to multiples of these ($.25 \times .2$ or $2.45 \div .05$).

Activities for Ratios (Objective K)

A *ratio* is a relationship between two quantities. For example, the ratio of girls to boys in Mrs. Smith's class is 17 to 21. Ratios may be expressed by separating the two quantities with a colon or as a fractional numeral.

17:21 or 17/21

- Have students represent ratios with objects such as different colored chips, beans and popsicle sticks, or characteristics of pupils. For example, ask them to represent ratios of 5:6 (blue chips to red chips), 8:5 (boys to girls), and 15:20 (beans to sticks). Have them write the ratios that you represent with materials or show on the overhead. Divide students into pairs and have them represent and write ratios for each other.

Activities for Proportions (Objective 17G–17H)

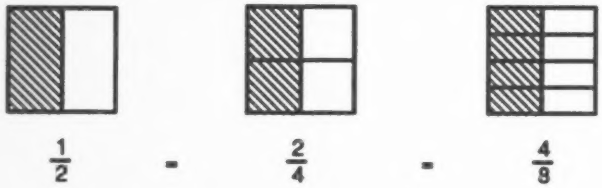
The concept of proportion is an extremely important applied mathematical idea, yet is very difficult for students through the upper elementary grades to master since they must focus simultaneously on two (equivalent) ratios or fractions—something that is difficult for concrete thinkers to do.

An example of a proportion would be: If 3/4 of the eighth grade students in Watkins Junior High watch MTV and there are 180 students in the eighth grade, how many watch MTV?

$$\frac{3}{4} = \frac{?}{180}$$

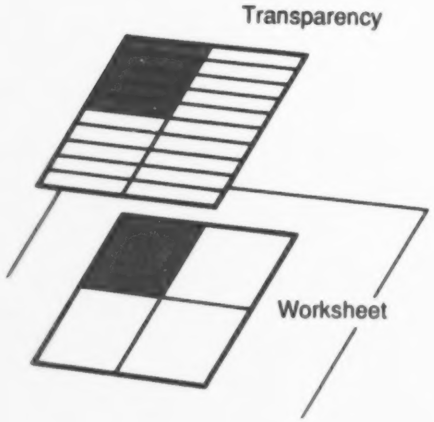
One way to develop the concept of proportion is to use manipulatives that represent two ratios or fractions in the same space (see Figure 14.11). In Chapter 12, we discuss developing the idea of equivalent fractions by dividing up a shaded region differently to show the two fractions. Here we do essentially the same thing, except that each fraction is represented with its own object and the two are placed together to show that the same portion of the whole is being considered.

FIGURE 14.11
Equivalent fractions.



One way to accomplish this is to represent one fraction on a sheet of paper and the other fraction—using the same size and shape for the "whole"—on a transparency (Figure 14.12). The transparency is then superimposed over the sheet to show that both shaded portions cover the same area. Another possibility is to use two different colors of transparencies on the overhead to demonstrate this concept to your class. Once you have represented a proportion, discuss the model in relationship to the symbolic representation for the proportion (Figure 14.13). Model and discuss several such proportions.

FIGURE 14.12
Using transparencies to model proportions.



Since these materials are somewhat difficult to make and manipulate, you may wish to forgo having students do any of the above practice activities with these materials. The elastic, discussed next, is better for manipulation by students. This other model for proportions consists of a meter/yardstick and an elastic band. Purchase 1-in. wide, 30-in. long braided elastic from the sewing department at a department or variety store. Stretch the elastic to its maximum and tack both ends to soft wood (e.g., a 2-by-4). Use the meterstick and a permanent black marker to mark 12 decimeters on the elastic (see Figure 14.14).

To model proportions with this manipulative, represent a ratio on the yardstick side of the ruler (you could use two colors of plastic tape or nonpermanent markers for this; for example, the ratio $^{5}/_{12}$ could be shown on the yardstick by a mark in blue from 0 to 5 and a mark in red from 0 to 12) and stretch the elastic so that the 0 on the elastic corresponds with the left side of the stick and the two denominators of the ratio correspond. The numerators of the proportion will also correspond.

FIGURE 14.13
The relationship between model and symbols for proportions.

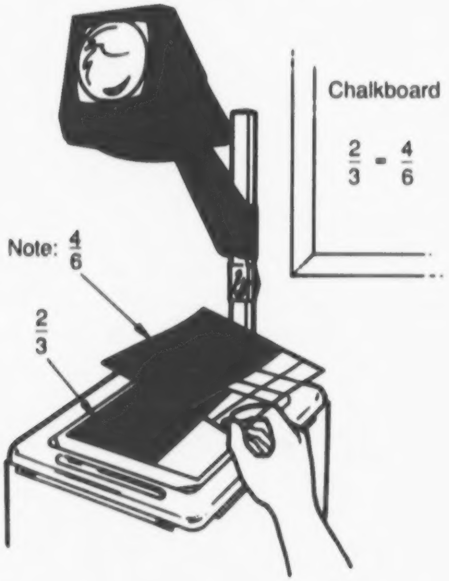

Chalkboard
2/3 = 4/6
Note: 4/6
2/3

FIGURE 14.14
Proportion elastic.

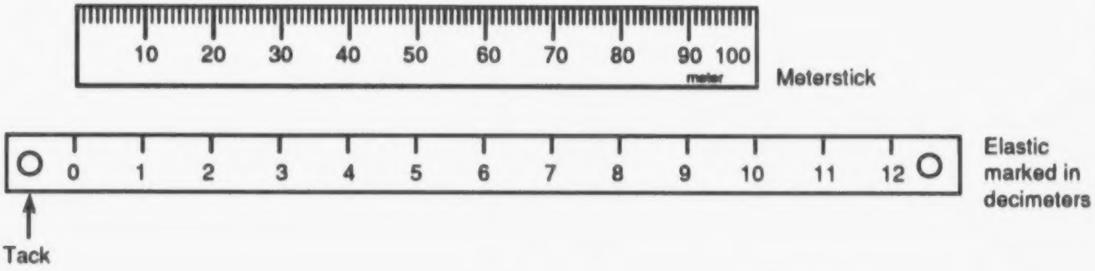
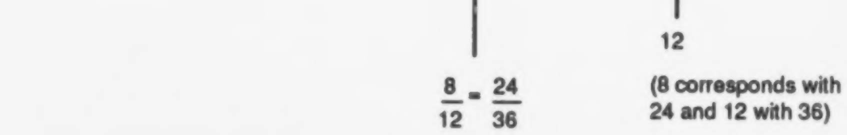

10 20 30 40 50 60 70 80 90 100
meter
Meterstick
0 1 2 3 4 5 6 7 8 9 10 11 12
Elastic marked in decimeters
Tack

FIGURE 14.15
Proportions using the elastic.


1 2 3 4 5 6 7 8 9 10 11 12
24
36 39
1 ft.
2 ft.
3 ft.
0 1 2 3 4 5 6 7 8 9 10 11 12
12
8/12 = 24/36
(8 corresponds with 24 and 12 with 36)

Represent and discuss several proportions using this model. Use proportions that correspond to the limitations of the materials such as $^{9}/_{36} = {}^{3}/_{12}$ and $^{6}/_{30} = {}^{2}/_{10}$ (the 10 on the elastic will correspond with the 30 on the yard stick). The use of this device for finding missing values in proportions is discussed below.

After students understand the concept of proportion, introduce a symbolic method for setting up and solving proportions. To set up a proportion, corresponding values are across from each other. In the example below, the 4 corresponds to the 180 because it represents the whole group. The 3 corresponds to the missing value—the part.

$$\frac{3}{4} = \frac{?}{180} \begin{matrix} \leftarrow \text{Part} \\ \leftarrow \text{Whole} \end{matrix}$$

To solve proportions, represent it in a way that isolates the missing value on one side of the proportion, and in the numerator. This can be accomplished by multiplying and/or dividing both sides of the proportion by the same number (the proportion does not change when you do this). For example:

PROBLEM: $\frac{3}{4} = \frac{x}{180}$

STEP 1: Multiply both sides by 180, cancel.

$$180 \times \frac{3}{4} = \frac{x}{\cancel{180}} \times \cancel{180}$$

STEP 2: Simplify.

$$\frac{\overset{45}{\cancel{180}} \times 3}{\cancel{4}} = x$$

$$135 = x$$

PROBLEM: $\frac{5}{18} = \frac{20}{x}$

STEP 1: Multiply both sides by x, cancel.

$$x \times \frac{5}{18} = \frac{20}{\cancel{x}} \times \cancel{x}$$

STEP 2: Multiply both sides by $^{1}/_{5}$, cancel.

$$x \times \frac{\cancel{5}}{18} \times \frac{1}{\cancel{5}} = 20 \times \frac{1}{5}$$

STEP 3: Multiply both sides by 18, cancel.

$$\frac{x}{\cancel{18}} \times \cancel{18} = \frac{20}{5} \times 18$$

STEP 4: Simplify.

$$x = \frac{\overset{4}{\cancel{20}}}{\cancel{5}} \times 18$$

$$x = 72$$

Later, you may wish to introduce the idea of *cross multiplying*. When you cross multiply, you multiply the extremes of the portion together and the means together. It is also possible to exchange extremes with each other and means with each other.

$$\begin{array}{lcccl} \text{Extreme} & 4 & & 20 & \text{Mean} \\ & & \times & & \\ \text{Mean} & 5 & & 25 & \text{Extreme} \end{array}$$

This is permissible because we can multiply both sides of an equation by the same value. The value that simplifies the fractions is the product of the denominators, as shown below.

$$\frac{a}{c} = \frac{b}{d}$$

$$(d \times c) \times \frac{a}{c} = \frac{b}{d} \times (d \times c)$$

Applying the commutative, associative, and unit properties, we may simplify:

$$(d \times \cancel{c}) \times \frac{a}{\cancel{c}} = \frac{b}{\cancel{d}} \times (\cancel{d} \times c) \rightarrow d \times a = b \times c$$

PROBLEM: $\frac{5}{18} = \frac{20}{x}$

STEP 1: Exchange extremes to put X in numerator.

$$\frac{5}{18} = \frac{20}{x} \qquad \left(\frac{\text{Extreme 1}}{\text{Mean 1}} = \frac{\text{Mean 2}}{\text{Extreme 2}}\right)$$

STEP 2: Put mean 1 in numerator with mean 2.

$$\frac{x}{18} = \frac{20}{5}$$

STEP 3: Simplify.

$$x = \frac{18 \times 20}{5} = 72$$

You may wish to have the students use calculators as for solving proportions.

CALCULATIONS WITH PERCENTAGES

Understanding Percentage Numerals (Objective N)

The topic of percentages is, sadly, among the most neglected in elementary mathematics. Yet, percentages are among the most used numerals in everyday life, especially in relationship to the commercial world. Research (e.g., Wiebe 1982; NAEP, 1983) has shown that many (if not the majority of) students at the upper elementary, high-school, and even college levels have major misunderstandings about percentages and are unable to solve problems involving them.

What are percentages? In one sense, they are fractions—43% means 43 out of 100:

43% → 43/100.

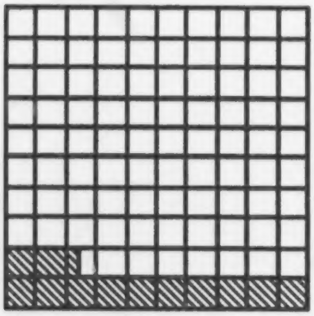
Thus, simply reading and understanding what a percentage numeral means are related to the student's knowledge about fractions. In reality, however, the idea of percentage is more complex. If we say that 47% of the households in California own two or more automobiles, we are not implying that there are only 100 households in California. Rather, we are reducing the fraction of households in California to a fraction with 100 as its denominator.

$$\frac{3,855,255}{8,567,234} = \frac{45}{100} = 45\%$$

The obvious reason for using percentages rather than fractions to express quantities is that, in most instances, the fraction would be too unwieldy. Consider, for example, the statement, "three million, eight hundred fifty-five thousand, two hundred fifty-five out of eight million, five hundred sixty-seven thousand, two hundred thirty-four of the households in California have two or more automobiles." Besides being an unwieldy mouthful or taking up too much space in print, it doesn't tell people what they actually want to know. Most people do not care what the actual number of California households is, or what the number of those households with more than one automobile is. What is important is the proportion. Fractions with 100 as denominators serve as a convenient standard for expressing such ideas.

Beginning activities involving percentages may simply involve developing an understanding of what the numerals mean. After discussing their meanings, show the students shaded squares (divided up into 100 smaller squares) that represent a variety of percentages. Next, have students shade squares to represent percentage numerals, and have them write the numerals that correspond with shaded squares. See Figure 14.16.

FIGURE 14.16
Shading squares to represent percentages.



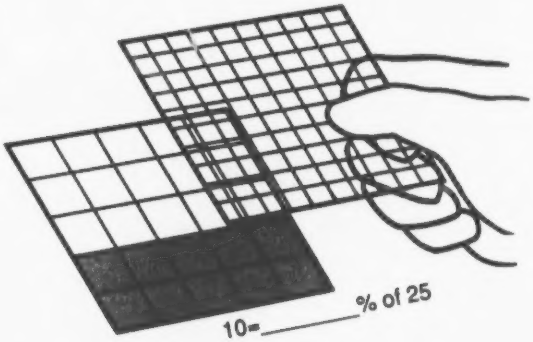
$12\frac{1}{2}\%$

Manipulative Activities for Percentages (Objectives O–P)

When developing students' understanding of percentage concepts, the most important idea is that of a proportion between the fraction expressed with the real values (e.g., the actual number of people supporting the president's policy over the total population), and the equivalent or proportional fraction with 100 as its denominator. The two models that can be used to help develop this understanding are extensions of the two discussed above for developing understanding of proportions.

■ One model for percentage operations consists of a transparency measuring 5 cm by 5 cm square but divided with an indelible marker into 100 small squares. The transparency is used on a worksheet of the same size, marked in 5 cm by 5 cm squares, a portion of which is shaded to represent sets of objects. See Figure 14.17.

FIGURE 14.17
Transparency and worksheet.



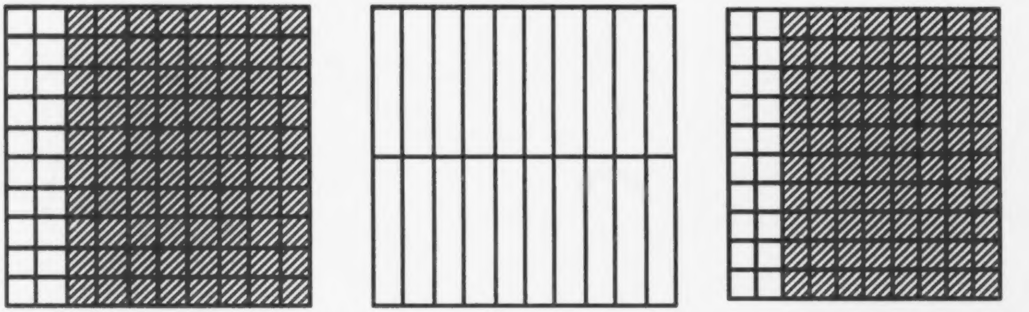
When using these materials to solve percentage problems, you may wish to re-express your percentage problems as proportions so that the manipulations and discussion correspond with those presented above in the discussion on proportions.

10 = ___ % of 25 → 10/25 = ?/100

The materials are manipulated as follows to use this model in teaching percentage operations:

1. *Problem:* 10 = _______% of 25. The 5 by 5 worksheet square has been pre-divided into 25 congruent parts and shaded to represent 10. Place the transparent percentage square on the worksheet square and count the number of percentage squares covering the shaded region. This is the missing value in the above problem.
2. *Problem:* 80% of 20 = _______. Since the percentage is given, the student shades the transparency to show 80% using an erasable transparency marker. Place the transparency on a worksheet square that has been pre-divided into 20 congruent parts and count the number of squares covered by the shaded area. See Figure 14.18.
3. Problems of the form "25% of _______ = 20" cannot be solved with this model.

When making worksheets containing squares and accompanying problems for students to solve, you may wish to set up only problems that require the worksheet squares to be divided into 5, 10, 20, 25, or 50 congruent parts. Not only will this ease your task of subdividing and drawing the squares, but it will allow you to construct the squares in such a way that each line on the worksheet square corresponds with a line on the transparency. Students, thus, will not have to estimate



Percentage transparency shaded to show 80%.

Worksheet square divided to show 20.

Percentage transparency superimposed on worksheet square (16 squares covered by shaded area).

FIGURE 14.18
Using the transparency to find 80% of 20.

what portion of squares are in the shaded region in order to find answers to problems.

The other model consists of the elastic discussed above, now marked from 0 to 100 with multiples of 10%, as in Figure 14.19.

To use this model, place the elastic next to the ruler so that the zero on the elastic corresponds with zero on the ruler. The 100% mark on the elastic corresponds with the *whole* on the ruler (in the general form of the problem, "% of whole = part"). The percentage value on the elastic will be aligned with the *part*. (See Figure 14.20.)

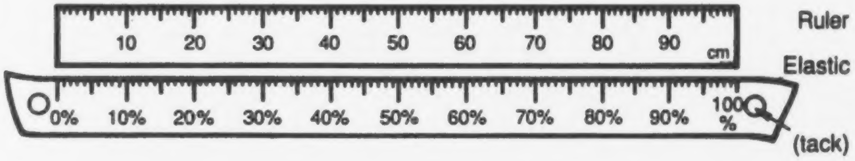

Ruler
10 20 30 40 50 60 70 80 90 cm
Elastic
0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%
(tack)

FIGURE 14.19
Percentage elastic.

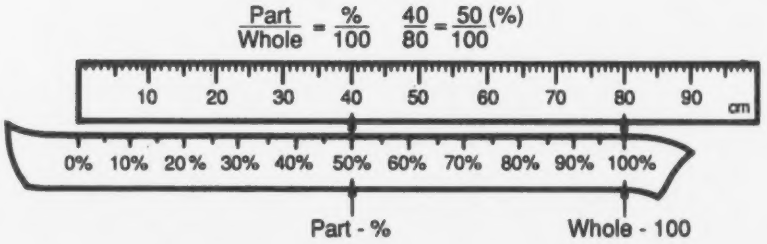

Part/Whole = %/100   40/80 = 50/100 (%)
10 20 30 40 50 60 70 80 90 cm
0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%
Part - %
Whole - 100

FIGURE 14.20
Using the percentage elastic.

The following problems illustrate the way the elastic is manipulated:

1. *Problem:* ________% of 80 = 16. Stretch the elastic and place it so that 0% is at the zero end of the meterstick and 100% on the elastic is aligned with 80 on the meterstick. Sixteen on the ruler is now aligned with the answer on the elastic—20%. See Figure 14.21.

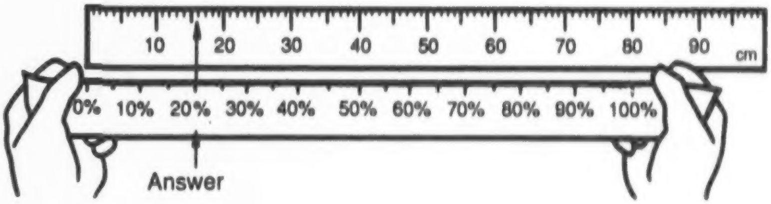

10
20
30
40
50
60
70
80
90
cm
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Answer

FIGURE 14.21
_____% of 80 = 16.

2. *Problem:* 70% of 30 = _______. Place 0% on the elastic at the zero end of the yardstick and stretch it so that 100% is aligned with the 30. The 70% mark will be aligned with the answer. See Figure 14.22.
3. *Problem:* 50% of _______ = 44. Place 0% of the elastic at the zero end of the meterstick and stretch it so that 50% is aligned with 44 on the ruler. The 100% mark on the elastic will be aligned with the answer. See Figure 14.23.

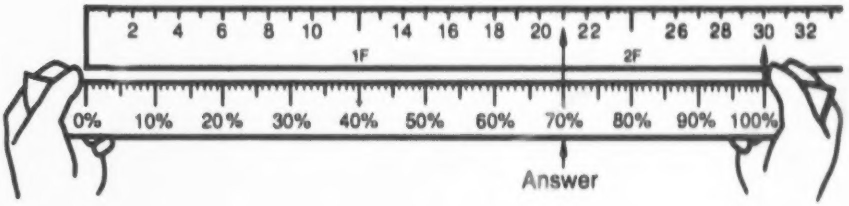

2
4
6
8
10
14
16
18
20
22
26
28
30
32
1F
2F
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Answer

FIGURE 14.22
70% of 30 = _____.

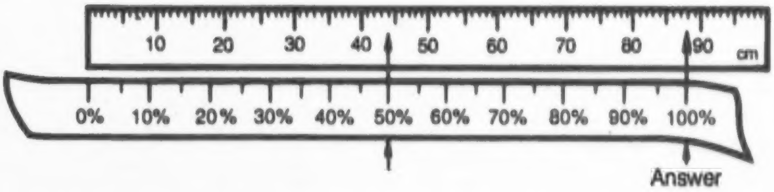

10
20
30
40
50
60
70
80
90
cm
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Answer

FIGURE 14.23
50% of _____ = 44.

When using these models, be sure to consider the limitations of the model. Teacher examples and student activities with them should be carefully planned. With the percentage elastic, for example, the *whole* in the problem should be between 60 and 100 cm if you are using the meterstick and between 25 and 39 if you are using the yardstick. Also, remember that the goal of the manipulations is to develop understanding. Accept answers obtained with the materials that are reasonably close to the exact answer.

Symbolic Activities for Teaching Percentage

A variety of techniques for solving percentage problems have been used in the schools. Perhaps the best and most meaningful of these is the proportions method. The first step in solving percentage problems with this technique is to convert the problems to proportions. One way to do this is to have students decide whether each of the values in the problem is one of three possibilities—the *percent*, the *part*, or the *whole*. Consider the following problem:

> There are 125 seventh graders at Roscoe Intermediate School. Seventy-five of these students, or 60%, went to the class dance.

In this problem, the *whole* (all the students) is 125, the *percent* is 80, and the *part* equals 75. In general, a proportion involving percentages looks like the following:

$$\frac{\text{Part}}{\text{Whole}} = \frac{\text{Percent}}{100}$$

Substituting values we have

$$\frac{75\text{ (part)}}{125\text{ (whole)}} = \frac{80\text{ (percent)}}{100}$$

- Have students interpret a variety of percentage problems with no missing values, similar to the one above, by placing the values into proportions. Also, give them problems in proportion form and have them write corresponding verbal problems.

- Next, give them problems to solve that have a missing value. First have them write the problems in proportion form, then have them use cross multiplication to solve for the missing values.

Discuss the algebraic form of percentage problems—A% × B = C (or, *percent of whole = part*)—and show its relationship to the proportion form:

$$\underset{\sim}{A\%} \text{ of } B = C \rightarrow \text{Percent of whole and part}$$

$$\frac{A\%}{100} = \frac{C}{B}$$

or

$$\frac{\text{Percent}}{100} = \frac{\text{Part}}{\text{Whole}}$$

Then, use the techniques for solving proportions to find the solutions. For example:

EXAMPLE 1: 25% of 60 =

$$\frac{x}{60} = \frac{25}{100}$$

$$x = \frac{60 \times 25}{100} = 15$$

EXAMPLE 2: $133\frac{1}{3}$% of ____ = 120

$$\frac{120}{x} = \frac{133\frac{1}{3}}{100}$$

$$\frac{120 \times 100}{133\frac{1}{3}^{*}} = x \qquad \left({}^{*}133\tfrac{1}{3} = \frac{400}{3}\right)$$

$$\frac{120 \times 100}{400/3} = x$$

$$120 \times 100 \times \frac{3}{400} = 90 = x$$

EXAMPLE 3: ____% of 46 = 23

$$\frac{23}{46} = \frac{x}{100}$$

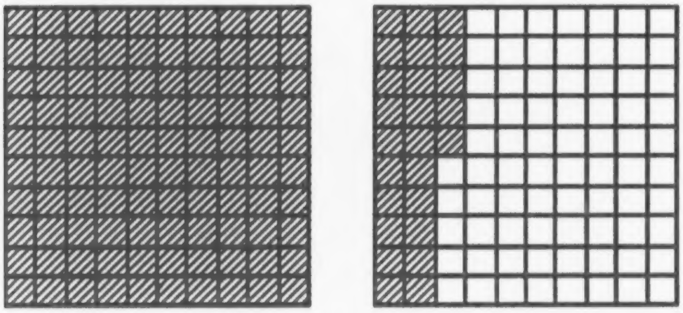
$$\frac{23 \times 100}{46} = x$$

$$50 = x$$

- Introduce students to models for mixed percentages and those greater than 100% through pictures. Example 2 above shows a mixed percentage and Figure 14.24 is a model for 125%. Have students shade shapes, and interpret shaded shapes, for mixed percentage numerals and those greater than 100%.

Problems involving mixed percentages and those greater than 100% are solved similarly to simple percentages. The trick is to simplify mixed fractions; Figure 14.24 shows one method for doing this.

FIGURE 14.24
A model for 125%.



125%

Many students have difficulty with percentage increase and decrease problems because they involve two steps. If you want to determine the sale price of a shirt marked $20.00 that is being discounted 25%, for example, you must first find 25% of $20.00, then subtract that value from the original price. When attacking such problems, many will correctly set up and solve the proportion, then state that the answer is the missing value in the proportion. To help them overcome such difficulties, stress the need to examine their answer to see if it is reasonable. Another possibility is to ask them to examine the problem before they calculate, then state the sequence of operations they will need to perform and to estimate an answer. Then, after they do their computations, have them compare their result to their estimate.

Calculators with % keys handle percentages in quite a variety of ways. Some convert decimals to percentages and vice versa. Most calculate percent decreases or increases. Some calculators use the following keystrokes to calculate a 25% increase over 120:

120 + 25 % (=)

Some calculators require you to hit the = button, others do not. Others use the following:

120 × 25 % +

Some use neither of these. It is recommended that you consult your owner's manual to see which technique your calculator uses.

Estimating Answers to Percentage Problems (Objective R)

Developing understanding of percentages through manipulative and meaningful symbolic activities is the biggest step in promoting students' ability to estimate. Encourage them to visualize the models with which they have worked. Give them numerous problems where they are asked to read and attempt to understand each problem, then estimate the answer. Have them use other techniques to compute the answer; then have them compare the estimated to the computed results and evaluate the accuracy of their estimate.

SAMPLE DRILL AND PRACTICE ACTIVITIES

Activity 14.1 Metric Addition

Objective: To add decimals.

Materials:

1. Meter stick
2. Rectangular objects
3. Pencil and paper

Methods:

1. If necessary, teach the students to find perimeters in metric units.
2. Have students measure perimeters of tables, sheets of paper, book covers, driver's licenses and so on and express the lengths in meters (e.g., the perimeter of a table might be .89 m + 1.456 m + .89 m + 1.456 m).

Variations:

1. Have them find differences in metric units—for example, how much taller is John than Susan? Again, express answers in meters, not centimeters, so that students must use decimals.
2. Have students find areas in metric units.
3. Have them find averages (e.g., batting averages), calculate kilometers per hour, kilometers per gallon, and so on.

Activity 14.2 Decimal Bingo

Objective: To add, subtract, multiply, or divide decimals.

Materials and Methods: See the discussion of math bingo in Chapter 20. Use decimal problems on the student worksheet.

Activity 14.3 Moral Puzzle

Objective: To solve the problems of the form ________% of B = C.

Materials and Methods: See Fig. 20.10.

SUGGESTED READINGS

Arithmetic Teacher 31(6), February 1984. Has several articles about decimals and percentages.

Dewar, J. M. Another look at the teaching of percent. *Arithmetic Teacher* 31(7), March 1984: 48–49.

Glatzer, D. J. Teaching percentage: Ideas and suggestions. *Arithmetic Teacher* 31(6), February 1984: 24–26.

Vance, J. H. An opinion poll: A percent activity for all students. *Mathematics for the Middle Grades (5–9)* (1982 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1982.

Wiebe, J. H. Manipulating Percentages. *Mathematics Teacher* 79(1), January 1986: 23–26.

________. Why do students have difficulty with percentage: A study of computational and error strategies. *Louisiana Educational Research Journal* 7(2), Summer 1982: 13–23.

QUESTIONS

1. Find the teacher's edition of a current sixth grade mathematics textbook. What suggestions are given for developing understanding of operations on decimals and percentages?
2. Describe a lesson involving estimation of products and quotients of decimals and calculators.
3. Refer back to the section of Chapter 11 (Multiplication of Whole Numbers) which discusses how grids and numeration blocks are used to develop understanding of two-digit multiplication. How could this model be used to develop understanding of multiplication of decimals?
4. Select 10 items from a department store (either visit the store or use a catalog) and write the name and price of each. Use either a calculator or a computer spreadsheet to find the price of each of these items at a 55% discount, a 25% discount, and a 10% increase in price. Record these values in a table.

CHAPTER 15

Measurement

Measurement is the attempt to quantify the world around us. There are hundreds of phenomena that we humans wish to quantify—density, length, acceleration, magnetic attraction, pressure, rate of flow, and time, to name only a few—but not many of these are taught in elementary mathematics. Those most commonly taught are money, time, length, area, volume, capacity, mass, and temperature. Each of these is related by a common concept: the fundamental idea of measurement, the *unit*. A unit of measurement is a distance, area, volume, capacity, etc., to which other distances, areas, volumes, capacities, etc., are compared.

MEASUREMENT SYSTEMS

Measurement goes back to before the beginnings of civilization. Early measurement units were derived from common objects or events. Length was measured by the span of the hand, the length of the thumb, or the length of a foot. Longer distances were often measured by comparing them to the distance a person could walk or ride in one day—"The village was three days walk away." Grain was measured by baskets and liquids by cups or barrels. The Egyptians, Babylonians, and Greeks all used systems of measurement derived, for the most part, from body units. One can imagine the disputes that arose in the marketplace when the buyer and seller each used different-sized units to measure the goods being purchased, each to his own advantage of course.

The Egyptians developed a fairly consistent, standard set of measures because of their need to determine property lines after annual floods; however, measuring devices and objects cut to "exact" measures (e.g., stones in the pyramids) from that period vary in length. It was not until the Middle Ages that attempts were made in Europe to *standardize* units of measurement. At one point, it was decided that the length of three barleycorns was to be an inch, twelve inches was a foot and three feet a yard. Still, all barleycorns are not exactly the same size, so King Henry I (1086–1135) of England declared that the distance from his nose to the end of his thumb would be one yard. King Edward I (1239–1307) established the first fixed unit when he declared that an iron rod would be the standard for one yard. In the 19th century, England, France, and the United States all established bureaus of standards to define and maintain fixed units of measurement. Each has its own standard units in platinum and kept under constant temperature to prevent variations.

The Customary System

Today, there are two major systems of measurement in the world: the English or U.S. customary system, used in the United States and a few third-world countries, and the metric system, used in the rest of the world. The U.S. customary system with its pounds, feet, miles, watts, and so on, evolved out of the ancient body-related units discussed above and the nonstandard English system used before American independence. The problem with this system is that it was not developed logically. There is no consistent or logical relationship between different units: 1 foot = 12 inches; 1 mile = 5280 feet; 1 cubic foot = 1728 cubic inches; 1 peck = 8 quarts, 537.605 cubic inches. In addition, many of the units in use in this system are defined differently for different types of measure. For example, a liquid quart is different from a dry quart, and a liquid ounce is different from a dry (avoirdupois) ounce, or from a troy (or apothecary's) ounce. Most people in the United States know little more than the basic units of measurement of length and capacity and do not know how to convert them to other less common U.S. customary units.

The Metric System

The metric system was developed to provide a more consistent, logical, and efficient standard of measurement. It was first proposed by French scientists in the 18th century and perfected by the International Bureau of Weights and Measures, established in 1875. It was adopted as one of the official systems of measurement in the United States in 1866. In 1975, the Metric Conversion Act was passed calling for voluntary conversion to the metric system. Although in everyday, practical situations the U.S. customary system is still used, scientists, the military, much of the automobile industry, many technological devices, and commercial items bought or sold internationally are measured in metric units.

The metric system has seven basic units: the meter for length, the second for time, the kilogram for mass or weight, the ampere for electric current, the mole for amount of substance, Kelvin for temperature, and candela for luminous intensity. From these, other units to measure area, volume, capacity, velocity, density, and power are derived. Rather than using the Kelvin scale which starts at *absolute zero*—the temperature so cold that atomic vibration ceases—we use the Celsius scale for everyday temperatures. This scale is related to the Kelvin scale, but sets 0 degrees as the freezing point of water and 100 degrees as the boiling point of water. The meter (a little longer than a yard), originally defined as one ten-millionth of the distance between the North Pole and the equator, is currently defined in relation light waves. Mass is similar to weight except that it does not vary with location; a mass of 5 grams is the same at the North Pole, the equator, or in space.

Metric prefixes indicating powers of ten are added to the base units to generate new units. For example, the prefix *milli* may be joined to the base *meter* to form the unit millimeter—one thousandth of a meter. Combining the prefix *kilo* with the unit for capacity, the *liter*, we have *kiloliter*—a thousand liters. Table 15.1 shows the more common metric prefixes.

The important bases, those relevant to everyday living and instruction in the elementary classroom, are shown in Table 15.2.

TABLE 15.1 Metric prefixes

Prefix	Value	Example
mega-	1,000,000	megagram
kilo-	1,000	kilogram
hecto-	100	hectogram
deka-	10	dekagram
(no prefix)	1	gram
deci-	.1	decigram
centi-	.01	centigram
milli-	.001	milligram
micro-	.000001	microgram

TABLE 15.2 Common metric units

Base	Type of measurement	Abbreviation
meter	length	m
gram	mass or weight	g
liter	capacity	l
square meter	area	m^2
cubic meter	volume	m^3

These basic units are all interrelated. A gram is defined as the mass of one cubic centimeter of water under specific conditions. A liter is the capacity of a container that will hold one cubic decimeter of material. One cubic decimeter of water has a mass of one kilogram. Try making conversions between cubic inches, cups, and pounds to appreciate the simplicity and logic of the metric system.

Even though everyone in countries where the official system of measurement is the metric system knows the eight prefixes listed in Table 15.1, only a few prefix-base combinations are used in everyday life. These, their meanings, and a practical reference to comparative units are listed in Table 15.3. If you know these, you will have no trouble at all getting along in countries where only the metric system is used.

TABLE 15.3 Commonly-used units of measure

Type	Unit	Meaning	References
Length	kilometer (km)	1000 meters	About .6 miles
Length	meter (m)	1 meter	One pace (39 in.)
Length	centimeter (cm)	.01 meters	Width of fingernail
Length	millimeter (mm)	.001 meters	Thickness of dime
Weight	kilogram (kg)	1000 grams	Medium-sized potato
Weight	gram (g)	1 gram	Raisin
Capacity	liter (l)	1 liter	Bottle of wine
Capacity	milliliter (mm)	.001 liter	Eyedropper of liquid

Learning the metric system is not difficult. All you need is some practical experience measuring a variety of objects with the units listed in Table 15.3. After that, you should learn the system of prefixes so that on the rare occasion that you encounter a measurement like a decigram, you will know what it means. You will also find it valuable to be able to convert from one unit with a given base to another within that same base, e.g., from millimeters to meters or kilometers. This type of conversion simply involves moving decimal points. Convert 25 centimeters to kilometers, for example, (since a kilometer is 1000 meters and a meter contains 100 centimeters), as follows:

$$1 \text{ cm} = .001 \times .01 = .00001 \text{ km}$$

and

$$25 \text{ cm} = 25 \times .00001 \text{ km} = 00025 \text{ km}$$

There is absolutely no reason to learn the conversion factors between metric and English units. The fact that these elaborate formulas and procedures have been taught in the past is partly responsible for the slow acceptance of the metric system in the United States and many people's fear of it. If ever you do need to convert from metric to U.S. units or vice versa, you will find the appropriate conversion tables in the nearest dictionary. Some calculators even have the conversion factors programmed in.

TEACHING MEASUREMENT TO CHILDREN

When teaching a particular kind of measurement to children, we should focus on developing understanding of the phenomenon we are trying to measure and the units we use to measure it. Units similar to those used by ancient mankind are most useful in developing an understanding of length, area, time, volume, capacity, and money. A sequence of instruction appropriate for teaching any type of measurement follows:

1. Use body units or other natural units to develop understanding of the *unit* concept and the phenomenon being measured;
2. Develop understanding of and proficiency with the standard units of measurement, including the ability to give reasonable estimates of measures using these units;
3. Do symbolic activities with the standard units, such as using formulas for computing areas or converting between metric units.

Stage three should be saved for older students who have completely mastered each unit, can use them accurately in measurement, and have the ability to estimate quantities with these units. This chapter concentrates on fostering the understanding of basic units of measurement.

A major question teachers should ask is *when* U.S. customary units should be taught. It is the author's opinion that because of its logical simplicity, its use in technology and science, its acceptance and use by most of the world and much of the U.S.A., and its being based on the decimal system of numeration, children should master the metric system before the U.S. system is introduced. Then, in the upper elementary grades the basic units (inch, foot, yard, mile, pound, quart, gallon) should be briefly introduced and experienced through measurement activities. Finally, the interrelationships among units within a type of measurement (e.g., units of length) should be presented and students should be shown how to use measurement tables and calculators to convert between units. Note, however, that if you decide to focus your instruction on U.S. units rather than metric units, the activities in this chapter could be used by substituting appropriate U.S. units for metric units.

Behavioral Objectives for Measurement

A. Name an object that has the approximate value of one cent, 5 cents, a dime, a quarter, one dollar, 5 dollars, 10 dollars, 100 dollars, 1000 dollars, and 10,000 dollars.

B. State and write the values of individual coins and bills and combinations of them and represent a written dollar/cents amount with the minimum number of bills and coins.

C. Solve problems involving money.

D. Name an event that occurs in a length of time of one second, one minute, one hour, one day, one week, one year.

E. Name the approximate amount of time (seconds, minutes, hours, days, weeks, months, or years) it takes for certain events to happen.

F. State the time from a digital clock.

G. State or write the time from a traditional clock.

H. State or write the day, month and year from a calendar.

I. Solve problems involving time.

J. Name an object that has a length of one millimeter, one centimeter, and one meter.

K. Measure to find lengths using millimeters, centimeters, or meters.

L. Estimate the length of an object using the appropriate metric unit.

M. Name a surface that has an area of one square millimeter, one square centimeter, one square decimeter, and one square meter.

N. Use tiling to find the area of a rectangular region.

O. Use a centimeter grid to find the area of a non-rectangular region.

P. Use measurement and a formula to find areas of regular shapes in square centimeters and square meters.

Q. Estimate the area of a region in cm^2, or m^2.

R. Use centimeter cubes to find the volume of a small rectangular container.

S. Use water displacement to find the volume of an object in cubic centimeters.

T. Use measurement and a formula to find the volume of a regularly-shaped rigid object in cm^3 or m^3.

U. Use a graduated container to measure a given quantity of liquid or the capacity of a small container in milliliters or liters.

V. Estimate the capacity of a container in milliliters or liters.

W. Name an object with a mass of about one gram or one kilogram.

X. Use a pan balance and metric weights to find the mass of an object in grams or kilograms.

Y. Estimate the mass of an object in grams or kilograms.

Z1. State the temperature of boiling water, freezing water, and room temperature in degrees Celsius.

Z2. Measure the temperature of a liquid or the air using a Celsius scale.

Z3. Estimate the temperature of a liquid or the air in degrees Celsius.
Z4. Convert from one metric unit to another of the same type of measurement.
Z5. Measure objects in U.S. customary units.
Z6. Estimate measures in U.S. customary units.

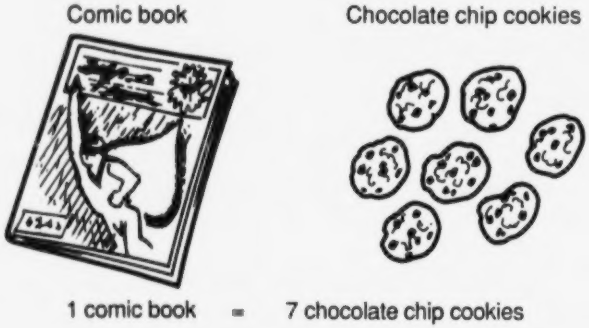
ACTIVITIES FOR TEACHING MEASUREMENT

Teaching Money (Behavioral Objectives A–E)

Money concepts should be developed similarly to other measurement concepts. First, introduce students to the idea of comparing one value to another value (the unit). This idea can be established by using nonstandard units of value known to and used by children, such as marbles or gumballs. The next step is to develop understanding of the usual money units of our society through much practical measurement experience. Next, after basic understandings about the basic units have been established, the teacher should attempt to interrelate the units. How many pennies are in a dime or how many quarters in a dollar? Finally, the teacher asks students to solve a variety of problems involving money that make use not only of their knowledge of money but also other areas of arithmetic.

- Select familiar objects of value to children, other than money, to use as units of value. Marbles, candy or comic books are examples. Compare other, more valuable objects to these "monetary" units (see Figure 15.1). Have students decide how much each of the more valuable items is worth in marble or candy money. You may wish to have an actual sale using this type of money. For example, you could bring apples or cookies to school and have students pay for them with marbles. These marbles could later be used in a variety of other math lab activities (see the weighing activities later in this chapter).

FIGURE 15.1
Measuring value.


Comic book
Chocolate chip cookies
1 comic book = 7 chocolate chip cookies

- Bring a jar of pennies to class and use them in discussions about and activities with money. Discuss how many pennies it would cost to buy a variety of items from around the classroom. Have the students play store where the only monetary unit used is the penny.

- Do similar activities with nickels, dimes, quarters and dollar bills. Play money can be used for the larger denominations of coins and bills.

- Make a playing board with spots for each child's savings account and a "bank" from which the players obtain money and make exchanges (see Figure 15.2). Players in turn roll a die and collect that amount in pennies from the "bank." Exchanges are made as needed, 5 pennies for a nickel, 2 nickels for a dime, and 2 dimes and a nickel for a quarter. The first player to get a quarter, 2 quarters, or a dollar (whatever is decided at the start of the game) is the winner.

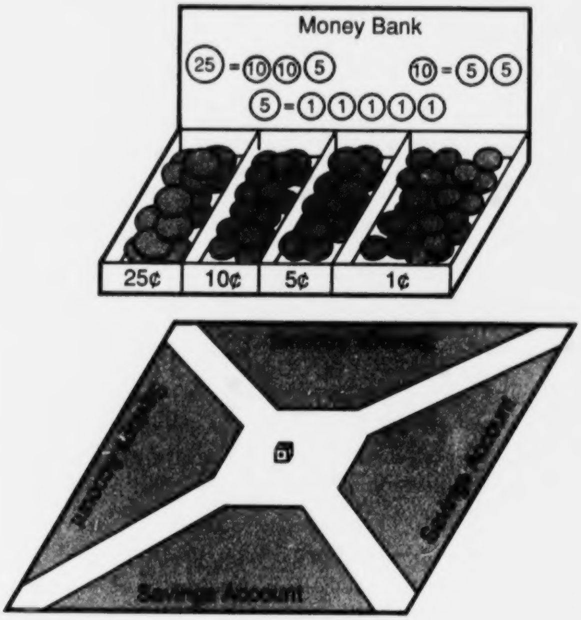


FIGURE 15.2
The playing board for "Money Bank."

- Do activities to help develop an understanding of the value of larger bills. Ask the students, for example, what they could buy with 1000 dollars. Use department store catalogs or other price sheets to help in this activity.

- Select pages from department store catalogs or advertisement supplements and have students select items, then represent the amount the item costs with real or play money. When working in a one-to-one or small-group diagnostic setting, use real money if possible.

- Make menu cards using pictures of food cut from magazines (sometimes it is possible to get children's menus from restaurants). Ask children to select items they would like for lunch and calculate the cost of a meal, using calculators.*

- Collect grocery ads from newspapers and have children make grocery lists and compute the total. Older children could plan a day's menu or a week's and calculate the cost.*

- Have students represent monetary values stated by you or found in a catalog using the minimum number of bills and coins. Use regrouping techniques to represent these values in different ways.

*Activities by Jewell Garner.

- Have students represent values like $3.79 with the minimum number of coins and bills. Have them find out how many other ways this amount can be represented. It might be helpful to have the students mark the possibilities on a chart with columns for pennies, nickels, quarters, and dollar bills. See Figure 15.3.

FIGURE 15.3
Problem solving with money.

Different ways to make $3.79

Dollars	Quarters	Dimes	Nickles	Pennies
3	3			4
3	2	2	1	4
3	2		5	4
Etc.				

- Give the student a certain number of coins and bills selected so that the set of money could be simplified (e.g., 15 pennies, 7 nickels, 8 quarters and 5 dollar bills). Ask the student to change the money to represent the same amount with the fewest possible bills and coins.

- Give the students problems to solve involving money. Have the students "buy" objects from a catalog using a single bill and have them calculate, with calculators, the change they would receive. Have them compute the amount to leave in a restaurant for a tip if the meal costs a given amount. Using newspaper advertisements, have them determine what they would buy if they had $100 to spend. Have them determine how much it would cost to buy a stereo outfit and build a set of shelves for it.

Teaching Time (Objectives D–I)

Our method of keeping track of time is partially dependent upon natural phenomena. For children, the most obvious unit of time is the *day*, or 24 hour period. Other units of time that can be experienced, observed, and grasped by children are the *second*, *minute*, *hour*, *week*, and *year*. As with other types of measurement, the process of teaching and understanding of our measures for clock time and calendar time involves these basic units: the second, minute, hour, day, week, and year. The concept of *month* should also be developed; this is, however, more difficult since it varies in length and since it is neither a multiple of a week nor a factor of a year.

One of the most important ways to develop a sense of clock time in children is to mention daily the time that certain events occur. For example, school begins at 8:30, lunch time begins at 11:30 and is over at 12:15, school is out at 2:20, and your favorite television program begins at 7:30 in the evening. After frequently being made aware of the time of day and the events that regularly occur at certain times, children begin to have some sense of the time of day.

Beginning instruction for time is designed to make children conscious of the passage of time and of possible units for measuring it. Rig a pendulum from the top of a cabinet or another high spot using a long piece of twine and a rock. Use one swing of the pendulum as a unit of time and compare this unit to other events. How many swings occur while a wind-up toy runs? John can hold his breath for how many swings? How many swings does it take for Sally to run from the classroom to the other side of the playground and back? Recess lasts how many swings?

- Use one flip of an hourglass egg timer as a unit of time and measure longer events using this unit. Math class lasts for how many flips? Can you determine the number of flips in a day?

- Develop concepts of *now, before,* and *after*. Ask,
  What are we doing *now*?
  What do we do *after* lunch?
  What do you do *before* school?
  Do you get dressed *before or after* breakfast?

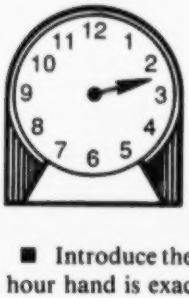
- Do activities to develop understanding of the *second*. Have students take turns doing a variety of activities like humming, walking, running, waving, turning the radio on, clapping their hands, etc. for one second. Have students name events that last for about one second. Turn a stop watch on for one second. Do similar activities but for multiples of one second. Hold your breath for five seconds, talk for 15 seconds, and so on. Do similar activities for one *minute* and multiples of one minute.

- An alarm clock or digital stop watch may be used to help develop meaning for the unit *hour*, or multiples of the hour. Set it at the beginning of math class to go off after the desired number of hours.

- Discuss events which take about one hour to occur, then discuss events which take 2 hours, 5 hours, and 10 hours to happen. Repeat for longer units of time. This would be a good place for integrated science units. For example, present a unit on the 17-year locust. Try to develop a feeling for the amount of time it takes for one of these cycles to occur.

- Relate earlier activities with small units of time to the clock. Find a clock with only an hour hand or purchase a used electric clock and remove the other two hands, as shown in Figure 15.4. Point out the position of the hand at various intervals throughout the day and discuss the time. For example, "It is almost 4 o'clock," or "it is half way between nine and ten o'clock." Point out and have students practice different ways of writing the hourly times.

FIGURE 15.4
The one-handed clock.



- Introduce the "long hand," or minute hand, investigating where it is when the hour hand is exactly on a number or at other positions in between. Present and practice reading time by the half hour. Have students write different times when shown clock faces at the hour or half past the hour. Do similar activities for time to the nearest five minute intervals and finally for time to the nearest minute.

- Provide magazine pictures showing people engaged in certain activities that occur at certain times of the day: sleeping, eating breakfast, walking to school, and so forth. Ask, "What time of day is it, morning, afternoon, night?"*

- Discuss events that occur at different times of the day. Include the distinction between A.M. and P.M.

- With a piece of tape mark where the hour hand is when the school day starts. At the end of the day mark where the hand now is. Ask, "How far did the hour hand move? How many hours have you been in school?" Mark where the minute hand is at the beginning of recess. At the end of the recess period, mark where the hand is. Ask, "How far did the minute hand move? How long was our recess? How many *minutes*?"*

- Have students solve a wide variety of problems involving times during the day. For example, "If it is 9:45, what time will it be in four hours?" Clocks whose time reading can be easily manipulated would be very useful for some children in such activities. In later stages, include problems that involve days, weeks, months, or years. For example, "How much time will go by between 4:30 P.M. on March 8 and 9:00 A.M. on March 16?" *The Guinness Book of World Records* would be a source of a variety of interesting problems about time and other units of measure.

- Introduce digital clocks and watches and relate the time on traditional analog clocks to digital clocks (Figure 15.5). Have students translate from digital to traditional time and vice versa.

- Calendar activities can be incorporated into the daily routine of the normal classroom—each day a new number is added to the calendar on the wall and the dates of major events are discussed when they occur. Discuss the child's routine and events in her life in relationship to the calendar. "What was the date of a week ago?" "What were you doing on July 4?" "Where will you be on September 24, 1995?"

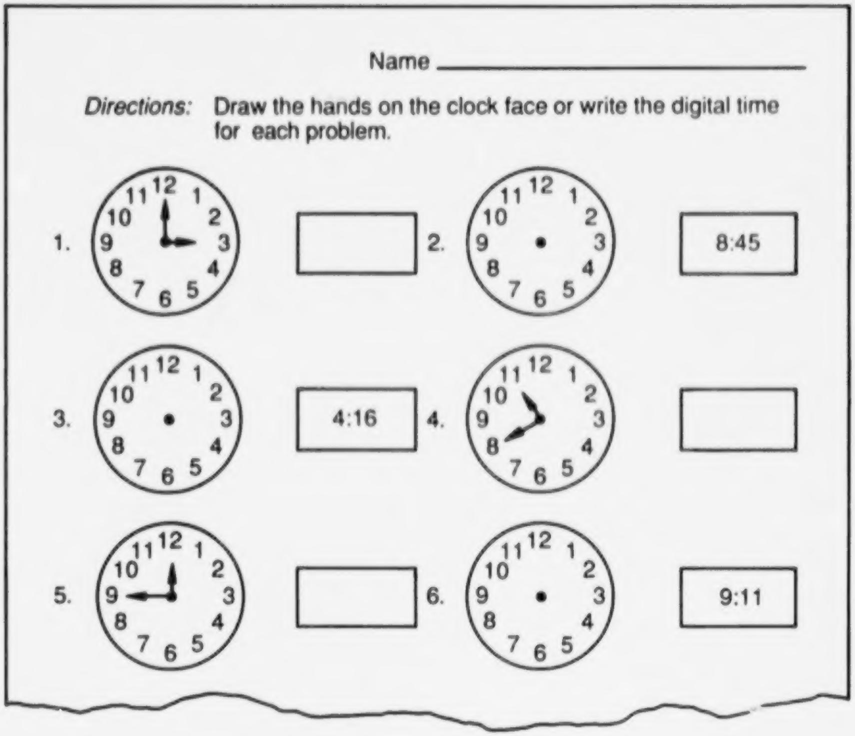
Teaching Measurement of Length (Objectives J–L)

The first step in developing understanding of linear measurement is to teach the *unit* concept. The easiest and most available units are *body units* like the span of the hand, the width of the thumb, and the pace. Have students measure a variety of lengths using these units (Figure 15.6). How wide is your desk? How wide is the playground? How wide is your math book?

- Take two objects that are identical except for length and ask children which is *longer*? *shorter*? Use pencils, crayons, chalk, etc. Add objects to the set that are the same length and establish that they are *equal* in length. Ask children to show you two objects which are *not equal*.*

- Give children sets of colored links or paper clips. Have them make chains and compare them. Who has the longest? the shortest? Are any equal in length? Have children work with a partner and make their two lengths equal. Establish that this can be done by adding links or by taking links away.*

*Activities by Jewell Garner.



Name ______________________

Directions: Draw the hands on the clock face or write the digital time for each problem.

FIGURE 15.5
A digital clock worksheet.

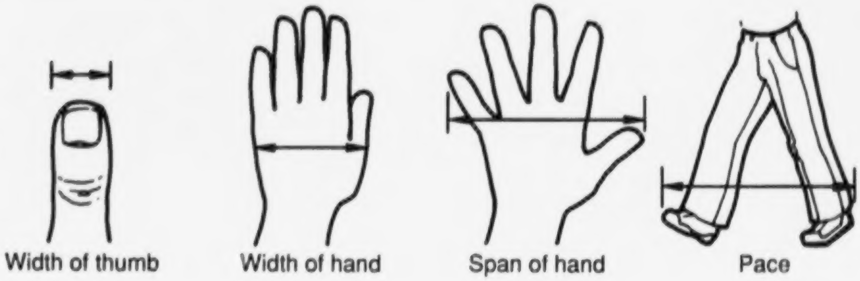


FIGURE 15.6
Using intuitive units for length.

- Have children work with a partner and compare hand spans. Whose handspan is longer, shorter, or are they the same? Compare children's spans with the teacher's. Measure the length of various items in handspans. Label them: for example, "The red book is 2 Terry handspans." "The bulletin board is "5 Teacher hand-spans."*

- Have children compare heights. Who is *tallest*? *shortest*? Are any the same? Have the group arrange themselves in order, from tallest to shortest.*

- Use interlocking blocks to make a measuring device for each child: for example, ten Unifix® cubes. Have children measure various objects, book shelves, tables, window sills, and so forth. Discuss the fact that if we all use the same unit to measure with it is much easier to tell how long an object is. Measure the objects labeled *handspans* and change the labels to ________ *blocks*.*

- Try to set up situations that will lead to the understanding of a need for a standard unit. How wide is this room using Tommy's pace? How wide is it using my pace? What can be done about this discrepancy?

- The orange and white Cuisenaire® rods or the units and tens rods in numeration block sets are good ways to introduce metric measurement through activities. Place the little white cubes side by side to measure lengths in increments of one centimeter. The long rods are one decimeter in length and can be used in combination with the centimeter rods to measure things like the width of a table top or the height of a person.

- Establish the relationship between these manipulatives and metric rulers and then have students do similar measurements with the rulers.

- Introduce the meterstick. Have the children experiment with the orange Cuisenaire rods to see how many are in a meter. Have the children identify objects around the room that are about a meter long. Have them do a variety of measurement with metersticks, expressing their answer to the nearest meter. Have students do more exact measurement, expressing answers as meters and centimeters (e.g., the length of the room is 8 meters, 54 centimeters).

- Prepare task cards on which each child, working with a partner, records his own body measurements in metric units.*

Arm length ________
Hand width ________
Thumb width ________
Length of little finger ________
Length of foot ________
Elbow to wrist ________
Knee to ankle ________
Waist to feet ________
Waist to top of head ________
Top of head to shoulder ________

- Make a metric ruler on a strip of paper and tape it to the wall and have the children measure their heights. List measurements in centimeters, then in decimeters and then in meters, as: 150 cm = 15 dm = 1.5 m.*

- Prepare task cards in which children use the meterstick to estimate, measure, and compute differences for such lengths as the distance between doors, and the length of blackboard.*

- Have students estimate lengths and then measure to see how close their estimates are.

- Point out the meaning of metric prefixes and the relationships between the different units for measuring length. Have students convert lengths in meters to lengths in centimeters, and vice versa.

- Introduce students to methods for estimating lengths. Perhaps the best technique is to compare the unknown distance to a known distance. For example, if an adult's stride is one meter, how many steps will it take the adult to walk across the playing field? If a football field is about 100 meters long, how many football fields (and, thus, how many meters) is it between two trees outside the school? If your hand is 10 cm across, how many hands are there in a sheet of paper? Have students estimate a variety of lengths in centimeters and meters.

*Activities by Jewell Garner.

Teaching Measurement of Area (Objectives M–Q)

Area is one of the less understood concepts we attempt to teach in the elementary grades, yet it is important to know in a large variety of jobs and in many practical, everyday situations. Perhaps one of the reasons many students have difficulties with the concept of area is that we do not spend enough time developing the *concept* of area and jump too quickly to *computations* with formulas for finding area.

When developing the concept of area, start with nonstandard, familiar units like *hands* or *stamps*; move to measurement activities with standard units and stay at this level until the student has an intuitive feeling for these units and is able to give reasonable estimates of areas; then do symbolic activities such as using formulas to find areas and converting between square centimeters and square decimeters.

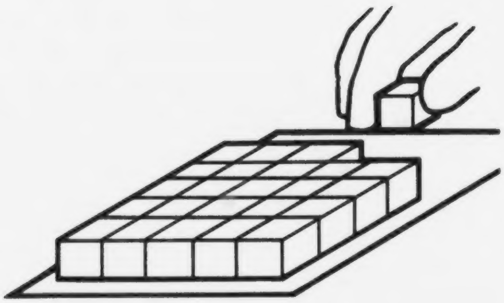
- Use familiar, reasonable objects as units of area. Although other shapes like triangles may be used, rectangles and squares are most reasonable for developing an understanding of units of area. Have students use their hands to find the area of several regions, like a desk top. Discuss the problems with using hands for this purpose. Next, measure the area of a table top using 8 ½ by 11 inch sheets of paper. Try to lead the student to find a way to deal with the portion of the papers that hang over the edge. (How about cutting those pieces off and putting the parts on the table together to make more wholes?)

- If the floor in your room is tiled, use the tiles to find the area of the floor in your room. If they are 12" squares, you can discuss the concept of one square foot.

- To find the area of smaller regions, use trading stamps. For example, have the student glue the stamps on a 3 by 5 card, then write its area in terms of these stamps as the measuring unit.

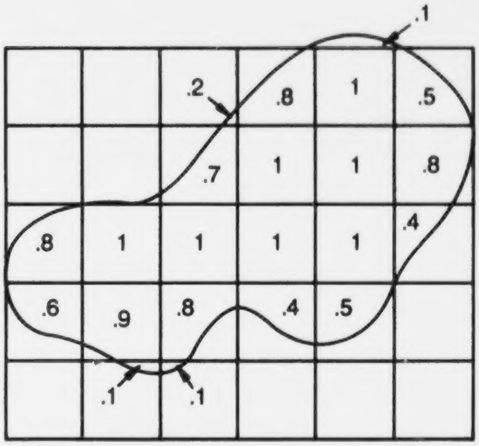
- Cuisenaire unit rods or the unit rods in numeration block sets may be used to develop understanding of the square centimeter. Find the area of small rectangles (a driver's license or a ticket to an athletic event for example) by finding out how many of the rods it takes to cover them. (Figure 15.7.)

FIGURE 15.7
Tiling to find area.



- Use the centimeter grid (Figure 15.8) to find the area of both regular shapes such as dollar bills, poker chips, and the bottom of soda cans, and irregular shapes such as the surface of one's hand (place the fingers together and draw around it).

FIGURE 15.8
Finding areas with centimeter grids.



Count the number of squares contained inside the enclosed region. The way you handle portions of square centimeters around the edges depends on the knowledge of the child with whom you are working. With younger children, have them imagine joining the partial squares together to form units (three parts approximately equal to a third of a unit square together form one unit). For students having background in decimals, have them estimate the portions of the units inside the region as decimals in tenths, then add them together.

- Introduce the square decimeter and the square meter and show students how to use measurement and the formula "1 × w" for finding areas. Have them measure a variety of rectangular regions and use the formula to find the area. Measure some of the regions that were measured using Cuisenaire rods or the centimeter grid paper and compare the answers obtained using the two methods. You may wish to have students use calculators to compute the areas.

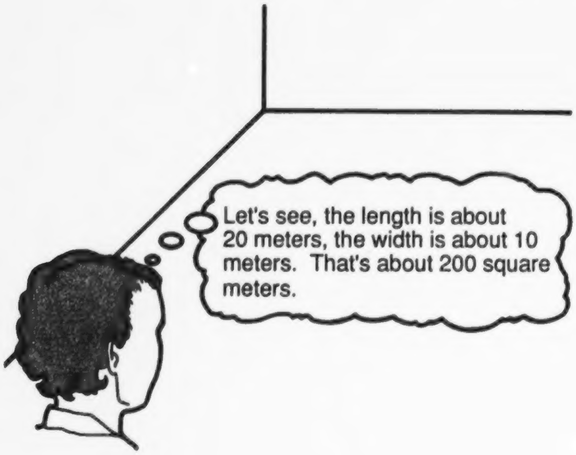
- Introduce other formulas for finding the area of other regions. Each time you introduce a new one, first have the student find the area of a shape (e.g., a triangle) using centimeter grid paper, then compare the answer obtained in that fashion to the answer obtained with the formula.

- Have the students estimate areas then measure to see how close they were to the actual area. One way to help children become proficient at estimating areas is to compare known areas to unknown areas. For example, if you know that the area of the top of a table is about one square meter, try to imagine how many of these will fit into the room. Another technique is to estimate the length and width of rectangular regions, then mentally multiply the dimensions to obtain an estimated area.

Teaching Volume (Objectives T–W)

Volume, like area, is not well understood by many people. A basic understanding that many people lack is of the unit concept for volume. We talk of units of volume but never give the learner a concrete example of these units, leaving it to their imagination to conjure up an impression of "one cubic inch" or "one cubic meter." Fortunately, standard manipulatives found in most elementary schools can be

FIGURE 15.9
Estimating areas.

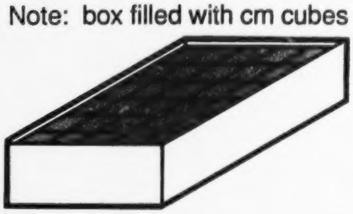

Let's see, the length is about 20 meters, the width is about 10 meters. That's about 200 square meters.

used as models. The white cube in Cuisenaire rod sets or the unit in numeration block sets may be used as models for the cubic centimeter. The large block in numeration block sets can be used as the cubic decimeter. Cubic meter kits are available from teaching material supply companies. Few teachers have the quantity of materials necessary to fill large boxes with cubic decimeters or rooms with cubic meters to demonstrate the meaning of these measurement ideas, but all children should have the experience of finding the area of two or three small boxes with cubic centimeters. An alternative is to use cubic inch blocks to develop the concept of measuring volume. After students have an idea of what measuring volume means, formulas can be introduced and used.

- Have students find the volume of rectangular containers using nonstandard units like dice, matchboxes, play blocks, or bricks.

- Collect small boxes of different sizes and number them. Have students fill them with cubic centimeters, as in Figure 15.10, and write their volume.

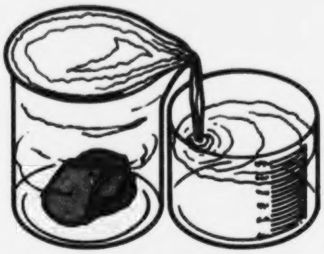
FIGURE 15.10
Using cubes to find volume.



- Introduce the formula for finding the volume of rectangular solids and use measurement (length, width, and height) to find the volume of the boxes used in the previous exercise. Have students compare the answers they obtained u[illegible] the two methods.

- To find the volume of irregularly shaped objects such as a rock, use water displacement. Nonporous objects that are heavier than water will displace their volume if placed in water. Thus, if you fill a container to the brim with wate[illegible] the object in the water, and capture the displaced water in a graduated beaker (see Figure 15.11), the amount of water in the beaker is equal to the volume of the object.

FIGURE 15.11
Using water displacement to find volume.



Note: rock in beaker full of water. Overflow captured by graduated beaker.

Another technique is to use a graduated beaker large enough to hold the object whose volume you wish to find, fill it to a given point with water, insert the object, record the new level of water, then subtract the original amount from the new to obtain the displacement.

- Have the students find the volume of rectangular objects (e.g., a stone cut in the shape of a rectangular solid, a wooden block—be sure to completely submerge the block to find its total volume) using water displacement. Then measure the objects' dimensions and multiply using the formula for finding volume to verify their findings.

- Introduce formulas for finding volumes of spheres and cones. Use measurement and the formulas to find the volume of several objects, such as a basketball, a small rubber ball, a wooden cone, or a soda can. You may wish to have the students check their answer to one or more of these by using water displacement.

- Have the students estimate volumes in cubic centimeters or cubic meters, then check to see how close their estimates are using measurement and formulas.

Teaching Capacity (Objectives U–V)

- Introduce the notion of capacity by using familiar containers like thimbles and drinking cups. Measure other familiar containers using these units. For example, you might experiment to find how many thimbles of water are in a can of soda or how many "cups" (nonstandard) are in a bucket.

- Provide children with a number of small containers and water or sand to experiment with. Ask, "Can these jars all hold the same amount? Which holds more, less? Do some hold the same amount? How many small jars of water does it take to fill this large one? If we pour the water from this jar into these three small ones is it the same amount of water or is it more or less?"*

- Use syringes (without the needle) with metric markings or other graduated containers to find the capacity of containers in milliliters. Use a one-liter soda pop bottle to find the capacity of larger containers.

- Have the students estimate capacities then measure to check the accuracy of their guesses.

Teaching Mass (Objectives W–Y)

- The nonstandard units of weight or mass used to develop an understanding of this concept could be dried beans, marbles, or play blocks. With a pan balance, find how many of these nonstandard units it takes to equal (i.e., to balance) familiar objects from around the room—a shoe, a book, a chalkboard eraser, a can of soda, or a rock.

- Prepare a set of objects that differ greatly in weight. Have children arrange the set from *heaviest* to *lightest*. As their skill in estimating increases, give the children objects in which differences in weight are smaller.

- Partially fill a box compactly with wooden blocks. Ask a child to lift the box. Now shake the box so the blocks are loose. Ask, "Is the box *heavier* now or is it *lighter*? Is it the same? How do you know?"*

- Introduce metric units of mass—the gram and kilogram. Using a pan balance, find objects that have a mass approximately equal to a gram. Find others that are approximately equal to a kilogram. (Figure 15.12.)

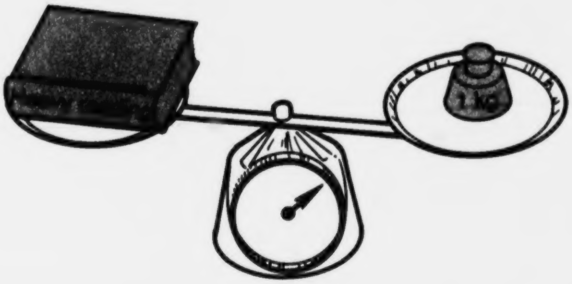


FIGURE 15.12
Finding mass using a pan balance.

- Use the pan balance to find the mass of the objects mentioned above in grams. If you have a metric bathroom scale available, have students find the mass of larger objects such as people, chairs, a bucket of water, or several large books in kilograms. If you only have available a bathroom scale that measures weight in pounds, take it apart and glue a metric dial over the pound display.

- Lead your students to discover the relationship between units of measure of capacity and mass. Have them find the mass of 45 ml of water, not including the container. Repeat for 200 ml and 1 liter. What pattern is evident?

- Have students hold various objects, estimate their mass, and then measure to determine the accuracy of their estimate.

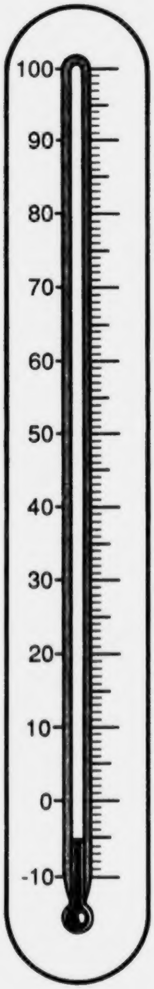
Teaching Temperature (Objectives Z1–Z3)

- Obtain a thermometer that measures temperatures below 0° and above 100° Celsius, as in Figure 15.13. Use it to measure the temperature of a bucket of ice water. If you have a refrigerator available, place a glass of water in the freezer section and measure the temperature at 1 minute intervals as it starts to freeze. Discuss the reasons why this point is not exactly zero on the Celsius scale.

---

*Activity by Jewell Garner.

FIGURE 15.13
A Celsius thermometer.



- Repeat with boiling water. Point out that the two beginning reference points on the Celsius scale are the freezing and boiling points of *pure* water at *sea level*.

- Measure the temperature of water and air under a variety of conditions. What is the temperature outside at daybreak? What is the temperature of water from the drinking fountain? At what temperature is a classroom comfortable? You may wish to have students graph variations in the outdoor temperature over a period of time.

- Have students learn a few basic reference points: in addition to 0° and 100°, 20° is room temperature, 38° is the normal body temperature of a human, – 20° is a cold winter day in Minnesota, and 42° is a hot summer afternoon in Phoenix.

- Have students estimate the temperature outside or in a room in degrees Celsius. Have them put their hand in a jar of water and estimate its temperature, then measure it with a thermometer to determine the accuracy of their guess.

Converting Between Metric Units (Objective Z4)

As mentioned above, to convert from one metric unit of length to another unit of length, you multiply or divide by a power of ten. In other words, you move the decimal to the right or left. To change 2.5 centimeters to meters, for example, since

1 cm = .01 m
2.5 cm = 2.5 × .01 m = .025 m

An algorithm for converting between units of a given type is to examine the list containing the prefixes between and including kilo- and milli-.
Kilo-
Hecto-
Deka-
(No prefix)
Deci-
Centi-
Milli-
To determine how far to move the decimal, count from the original unit to the new unit. For example, to convert 25 kilometers to decimeters, you move the decimal 4 places. Since you are changing from a larger unit to a smaller unit, move the decimal to the right. The sequence below demonstrates the use of this technique for converting 5.76 meters to millimeters and 2.6 centimeters to dekameters:

PROBLEM: 5.76 meters = ? millimeters

Kilometer
Hectometer
Dekameter
Meter
Decimeter
Centimeter
Millimeter

Down 3 places,
move decimal
3 places to right
5.76m = 5760 mm

PROBLEM: 2.6 cm = ______ dkm

Kilometer
Hectometer
Dekameter
Meter
Decimeter
Centimeter
Millimeter

Up 3 places,
move decimal
3 places to left

2.6 cm = .0026 dkm

- After appropriate instruction, give students a variety of practice problems.

U.S. Customary Units (Objectives Z5–Z6)

- Introduce students to the foot, inch, yard, and mile. Do a variety of measurement activities with the foot, inch, and yard. Take a one-mile walk to help develop an understanding of the mile. Have the children ask a parent to explain how the odometer in the car works and have them watch it the next time they are in the car. Have them estimate distances in miles to landmarks ahead of them, then check their estimates using the odometer.

- Introduce the pint, quart, and gallon. Use plastic gallon water or milk bottles to determine the capacity of larger containers. Repeat for the other units. Have the children estimate the capacities of containers using pints, quarts, and gallons.

- Have the children measure the weight of objects in pounds. Have them estimate weights, then use a scale to check their estimates.

SUGGESTED READINGS

Bradford, J. W. Making sense out of dollars and cents. *Arithmetic Teacher* 27(7), March 1980: 44–46.

Daane, C. J. Primary coin activity cards. *Arithmetic Teacher* 27(6), February 1980: 34–36.

Elliott, P. A. Money changer's bingo. *Arithmetic Teacher* 27(1), September 1979: 50–51.

Hart, K. Which comes first—length, area, or volume? *Arithmetic Teacher* 31(9), May 1984: 16–18 and 26–27.

Hiebert, J. Why do some children have trouble learning measurement concepts? *Arithmetic Teacher* 31(7), March 1984: 19–24.

Krustchinsky, R. Eekorian encounter—a measurement mystery. *Arithmetic Teacher* 31(7), March 1984: 14.

Nelson, G. Teaching time-telling. *Arithmetic Teacher* 29(9), May 1982: 31–34.

Riley, J. E. It's about time. *Arithmetic Teacher* 28(2), October 1980: 12–14.

Thompson, C. S., and Van de Walle, J. A single-handed approach to telling time. *Arithmetic Teacher* 28(8), April 1981: 4–9.

QUESTIONS

1. Determine and record in the chart in Figure 15.14 the conversion factors between the basic units of length in the English system (inch, foot, yard, rod, and mile). A calculator and a measurement table from a dictionary will be very useful for this task. Repeat for basic units of length in the metric system. How do the two systems compare?
2. Discuss the possible impact of the digital clock on the way we teach time.
3. Discuss a lesson designed to develop understanding of the concept of area using trading stamps.
4. Investigate at what point current mathematics textbook series start teaching the concept of volume. What approaches do they use?

FIGURE 15.14
A conversion activity.

	Inch	Foot	Yard	Rod	Mile
Inch	1				
Foot	12*	1		.06$\overline{06}$**	
Yard			1		
Rod				1	
Mile		5280			1

* 1 foot = 12 inches
** 1 foot = .0606 rods

	Centimeter	Decimeter	Meter	Dekameter	Hectometer	Kilometer
Centimeter	1		.01*			
Decimeter		1				
Meter	100		1			
Dekameter				1		
Hectometer					1	
Kilometer			1000**			1

* 1 centimeter = .01 meters
** 1 kilometer = 1000 meters

CHAPTER

16 Geometry

GEOMETRY IN THE CURRICULUM

Shapes are everywhere. Our environment is made up of planes, spheres, cylinders, rectangular prisms, triangles, cubes, circles, and lines. One of the first geometric experiences with mathematics that children have is when we help them pick out figures from their environment such as *circles* and *squares*. Later, if they study more geometry, they may be asked to find the surface area of a sphere or construct a perpendicular from a point to a line using a compass and straightedge. Geometry, as do other areas of mathematics, starts with concrete experiences with the environment and moves to abstract and symbolic experiences.

In teaching geometry, as in teaching other mathematical areas, we use models to help children understand abstract concepts. The idea that a plane has no thickness, for example, is difficult for a young child to grasp. However, when represented with a series of sheets of material like paper, each thinner than the previous, children can begin to comprehend. Finally, we can ask the children to imagine sheets of paper that are even thinner than the thinnest that we have shown them. By arranging physical objects properly and asking children to extend these experiences into the imaginary, we are teaching them to mentally manipulate their environment and, eventually, ideas—a major goal of schooling.

Technology, especially the computer, is important in the study of geometry. Children as young as kindergarten can discover geometric ideas through the use of Turtle Graphics in Logo.

It used to be that geometry was a subject saved for high school. We asked elementary school-aged children to define and find the areas of shapes like triangles and trapezoids, but we did not attempt to teach them the more formal, abstract aspects of geometry. Most mathematics educators now feel that some exposure to the more formal aspects of geometry and mathematics in the elementary grades is important. The abstractions learned will give them powerful tools for thinking, and for understanding and manipulating ideas. Furthermore, if children are only exposed to aspects of math related to computation, they are totally unprepared for the types of experiences that they encounter in high school mathematics. Finally, they get a false idea that mathematics involves only ideas related to computation rather than the more powerful ideas related to shapes, quantities, and numerical patterns.

Behavioral Objectives for Geometry

A. Identify shapes in the environment: circle, square, rectangle, triangle, parallelogram, and trapezoid.

B. Identify three-dimensional shapes in the environment: sphere (ball), pyramid, cylinder, cube, rectangular prism (box), and cone.

C. Use objects to demonstrate *over, under, far, near, bottom, middle, top, right, left, above, below, under, in front of, behind, beside, by, next to, between, inside,* and *outside.*

D. Define a point, line, ray, and line segment.

E. Define an angle.

F. Draw an angle of a given number of degrees.

G. Define a circle.

H. Draw a circle with a given radius.

I. Find the circumference of a circle with a given radius.

J. Define a triangle.

K. Identify an obtuse, acute, right, isosceles, and equilateral triangle.

L. State the relationship between the sides and hypotenuse of a right triangle (Pythagorean Theorem).

M. Define a plane.

N. Distinguish between regions and nonregions.

O. Find the number of points, segments, and regions in networks.

P. Identify parallel and intersecting lines.

Q. Define square and rectangle.

R. Find areas of triangles and rectangles.

S. Identify regular and irregular pentagons, hexagons, and octagons.

T. Identify trapezoid, parallelogram, and rhombus.

U. Define sphere, cone, cylinder, rectangular prism, and cube.

V. Find the surface area and volume of a sphere, cone, cylinder, rectangular prism, and cube.

W. Identify congruent and similar shapes.

X. Identify shapes that have point symmetry and line symmetry.

Y. Identify shapes that are translations of other shapes.

TEACHING GEOMETRY

Activities Involving Shapes in the Environment (Objectives A–C)

When children first come to school, they will already be able to use some basic geometric terms like *square* and *circle*. The problem is that they will probably be using these terms incorrectly. A *square* is a closed figure made up of four line segments of equal length, in a plane where the opposite sides are parallel, and all four angles measure 90 degrees (or a parallelogram with adjacent equal sides and a right angle); a *circle* is the locus of points equidistant from a point (in a plane). Children will use "circle" to refer to anything that is "round," including a basketball (a sphere), a disk of paper, a ring, or a tire. To them, a "square" might be a television screen or a sheet of paper (really, a thin rectangular solid). Although it is all right to allow some latitude when naming objects—the clock on the wall might, tentatively, be accepted as an example of a circle—we need to steer children toward more precise and correct definitions, models, and examples of geometric concepts. We need to give them a variety of concrete experiences that lead to an understanding of geometric concepts.

■ Bend stiff wire from coat hangers to form triangles, squares, rectangles, and circles. Discuss the attributes of these shapes. Note the number of angles and sides of each.

■ For older students, repeat the exercise above for parallelograms, trapezoids, rombuses, pentagons, hexagons, and so forth.

■ Give students sets of straws and pipe cleaners. Have them insert the pipe cleaners inside the straws and bend them to form the above shapes. This could be done in small groups.

■ Go on a geometry field trip. Walk around the school grounds, through the woods, or through the neighborhood and have students identify and record the different shapes they see. If students have been exposed to data-base programs, have them make a data base to record the shapes. Either allow the students to design the form of the data base or set up the form yourself as shown in Table 16-1.

TABLE 16.1 Data base for geometry field trip

	Field 1	Field 2	Field 3	Field 4	Field 5
Rec #	Object	Shape	Color	Where Found	Found by
1	Byc. Wheel	Circle	Silver	Schoolyard	Jenny
2	Book Cvr	Rectangle	Red, Gr.	Classroom	June
3	Ripples	Circle	Clear	Jones' Pond	Junito
etc.					

■ Introduce students to the difference between two-dimensional figures (e.g., circle) and their three-dimensional counterparts (e.g., sphere). Present numerous examples for each of the following three-dimensional examples: cubes, spheres, cones, cylinders, rectangular prisms, nonrectangular prisms. Have students bring other examples of the objects into school. Put the student's name on the object and place all objects that belong together next to a label (e.g., all cones or all cylinders).

■ Add three-dimensional shapes to the data base, or create a new data base.

■ To teach adverbs of relationship (over, under, etc.), use familiar objects like books, tables, boxes, balls, pencils, and children. First demonstrate the relationship for the students to see. Next ask children to demonstrate the relationship for you and the class using the appropriate objects. For example, "Sally, hold the pencil over the table," or "Jimmy, put the pencil between the book and the wall."

■ Place objects in a specific relationship (e.g., put a marble inside a cup), and ask the children to state the relationship ("Jose, where is the marble?").

Activities with Points, Lines, and Angles (Objectives D–F)

Since *points* and *lines* are abstractions (a *point* has no dimensions and a *line* has one—length) and do not have physical embodiments, it is necessary to use models to help children understand these concepts. A model for a point would be an extremely small speck of dust.

■ Present students with several models for points like a pin prick on a piece of paper, a point of lead on a sheet of paper, a star. Discuss why these are not points. Ask the students to suggest models for points. Have them imagine shrinking the smallest of these (the speck of dust) smaller, smaller, smaller until there is nothing left.

■ Ask students to write a LOGO procedure to draw a "point." Extend the activity by asking students to write a procedure to draw stars all over the screen at random.

■ Do activities similar to those above with lines. Present models for lines—a length of string pulled tight, a telephone wire stretched between two poles, the trail left by a jet plane, a line drawn on paper, a rubber band stretched tight. Discuss why these are not lines. Ask students to suggest other models for lines. Ask them to imagine a rubber band that could be stretched infinitely in two directions until it was infinitely long and infinitely thin.

■ Discuss the idea of line segment—part of a straight line bounded by two points. Typically, line segments are symbolized by AB where A and B are the endpoints. Have the students write a Logo procedure to draw a line segment, complete with the endpoints.

■ Present the idea of a ray—a line terminated at one end by a point and infinite in the other direction (Figure 16.1). Models would include a beam from a small flashlight or the flight of an arrow.

■ Introduce the idea of an angle—two rays with a common vertex (point). Present the idea of a degree and show students a little wedge measuring one degree so that they will understand the concept of *degree*. On the overhead projector, measure the number of degrees of a small angle (8 degrees or so) using the unit degrees as in Figure 16.2.

■ Make a set of angle measuring units, or angle checkers, including 1 ninety-degree angle, 1 forty-five, 4 ten, 2 five, and 3 two-degree angles as in Figure 16.2, out of tag-

FIGURE 16.1
Points, line segments, rays, angles.

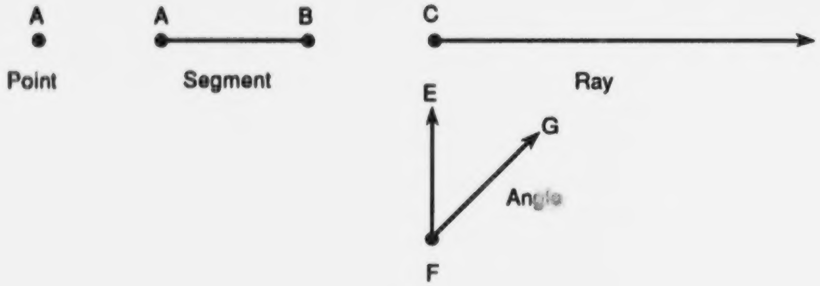
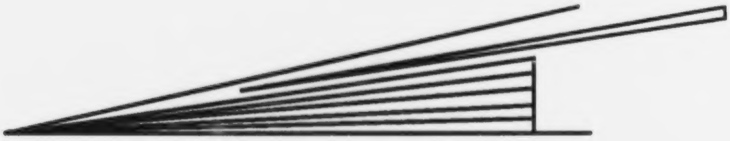


FIGURE 16.2
Measuring angles using angle units.



board or construction paper (Figure 16.3). Have students place them side by side inside to measure the size of angles on worksheets.

- Introduce them to a protractor and its use in measuring the size of angles. Have them practice using the device to find the measure of sizes of angles.

- Introduce them to the concept of *exterior angle* (see Figure 16.4). Use techniques similar to the above for measuring the exterior angle of various angles. Similarly, introduce them to the idea of the supplement of an angle (a useful idea when drawing triangles in Logo).

- Give examples and discuss acute angles (those less than 90°), right angles (90° angles), and obtuse angles (those greater than 90° but less than 180°). Have students classify angles as acute, right, or obtuse. Have them draw examples of each using Logo graphics.

FIGURE 16.3
Unit angles.

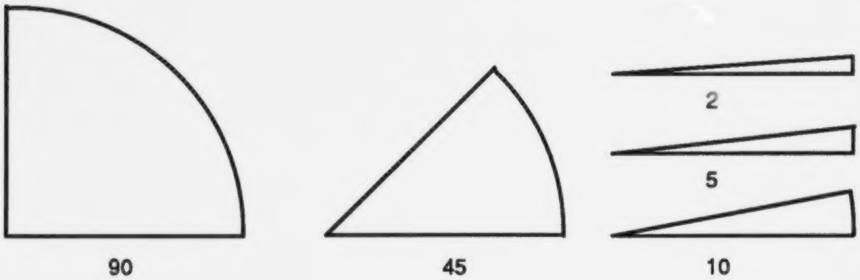
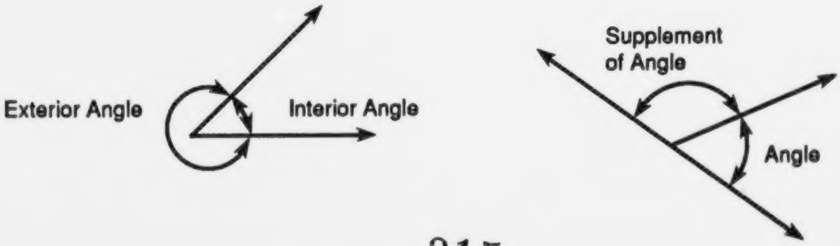


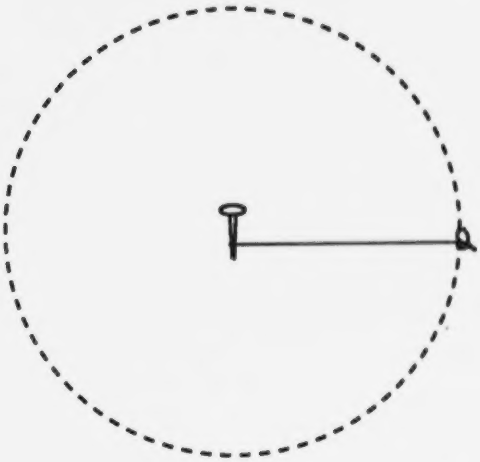
FIGURE 16.4
Exterior angles.



Activities for Circles (Objectives H–I)

The idea that a circle is series of points that are all the same distance away from a point (the center of the circle) can be conveyed using a nail, a piece of string, and a piece of chalk, as illustrated in Figure 16.6. Hammer the nail into the center of a 4 by 4 foot piece of plywood. Tie one end of the string to the nail and the other end to the piece of chalk (about 1½ feet away). An alternative is to tie one end of the string to a pencil and the other to a piece of chalk and have someone hold the eraser end of the pencil tightly to a spot in the middle of the blackboard. Make a series of dots with the string stretched tight. Ask the students what figure will be made if you make more dots. Continue making more and more dots until the circle starts to appear. Finally, draw the full circle (Figure 16.5).

FIGURE 16.5
Introducing the circle.



- Have children draw their own circles with a tack, string and pencil.

- Have children experiment with the REPEAT command in Logo and see if they can come up with a procedure to draw circles. Be sure to discuss, after they have obtained what looks like a circle, the difference between what they have created (usually, a regular polygon with 20 or more sides) and a circle.

- Give students a worksheet with a sequence of circles between 3 and 8 centimeters in diameter. Have them find the perimeter of the circles using heavy string (have them place the string around the circle and mark or cut the string to show the length of the perimeter, then measure its length in centimeters). Have them measure the diameter. Finally, ask them to divide the perimeter by the diameter using a calculator. After they have completed all the circles, ask them to look for a pattern. Many children will be able to discover the value of pi in this manner. (See Figure 2.9.)

- Teach children to use a compass to draw a circle with a given radius.

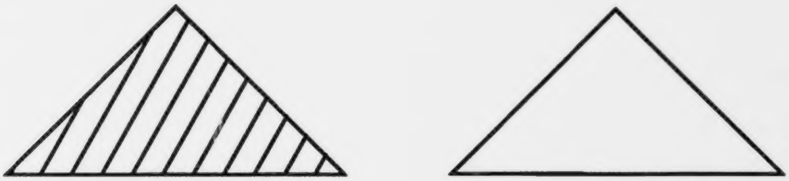
- Ask them to draw a circle with a given circumference.

Activities for Triangles (Objectives J–L)

The concept of *triangle* is not difficult to learn: most children can identify examples of triangles in the environment before they come to kindergarten. The more precise meaning (a figure made by connecting three noncollinear points with line segments) can be taught through a sequence of experiences using models.

- Cut a large acute triangle out of tagboard. Use wire, straws and pipe cleaners, wooden dowels, or other materials to make a triangle that is the same shape and size as the tagboard triangle but consists only of sides and, perhaps, vertices, as shown in Figure 16.6. Discuss the difference between the two. Make sure to finish the discussion by stressing that a triangle consists only of the vertices and the lines connecting the points. For the tagboard model, only the edges of the cardboard constitute the triangle.

FIGURE 16.6
Developing the concept of the triangle.



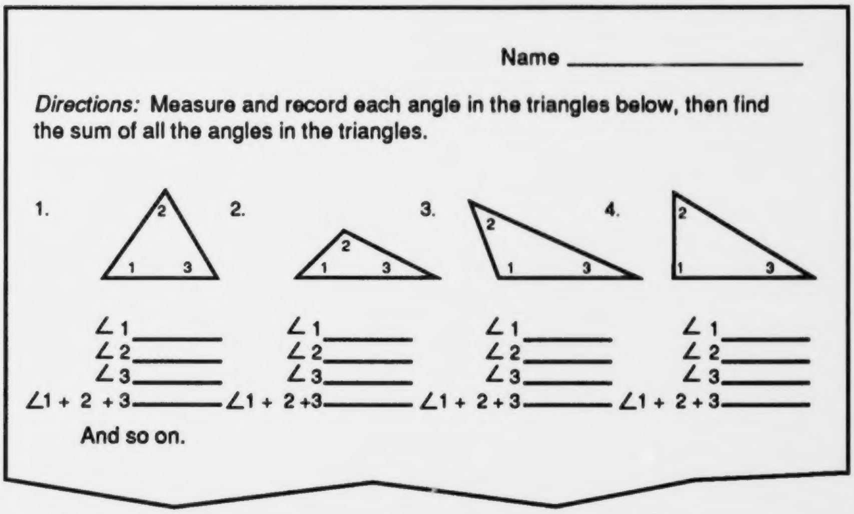
Tagboard triangle

Triangle made from wire

- Present students with a model for a triangle that includes the points as well as the connecting lines. For example, use little balls of modeling clay for the vertices and stiff wires to represent the line segments that join the vertices. Have children make their own triangles using balls of clay and toothpicks.

- Give students a worksheet, as in Figure 16.7, with several triangles. Have them measure using angle checkers, as discussed above, or protractors, record each angle of every triangle on the worksheet, and then find the sum of the angles for each triangle. Ask them to make a generalization about the sum of angles of triangles.

FIGURE 16.7
Discovering the sum of angles of triangles.



Name ______________

Directions: Measure and record each angle in the triangles below, then find the sum of all the angles in the triangles.

1.
2
1
3
2.
2
1
3
3.
2
1
3
4.
2
1
3

∠1 ______	∠1 ______	∠1 ______	∠1 ______
∠2 ______	∠2 ______	∠2 ______	∠2 ______
∠3 ______	∠3 ______	∠3 ______	∠3 ______
∠1 + 2 + 3 ______	∠1 + 2 +3 ______	∠1 + 2 + 3 ______	∠1 + 2 + 3 ______

And so on.

- Introduce the concept of right angle. Have students cut the corner off a sheet of notebook paper to use as their "right angle checker." Have them use protractors to measure the number of degrees in their angle checkers. Have them use the angle checker to find right angles around the room and record each of them on the board along with the name of the student who found it.

- Review the idea of obtuse and acute angles using angle checkers. Give students a worksheet with 5 or 6 triangles of different size and shape. Have them use their angle checkers to find and label all acute angles (label them with an *A*), obtuse angles (*O*) and right angles (*R*). Introduce the definitions for acute, obtuse, and right triangles and have the students classify each triangle on the worksheet with one of the three labels.

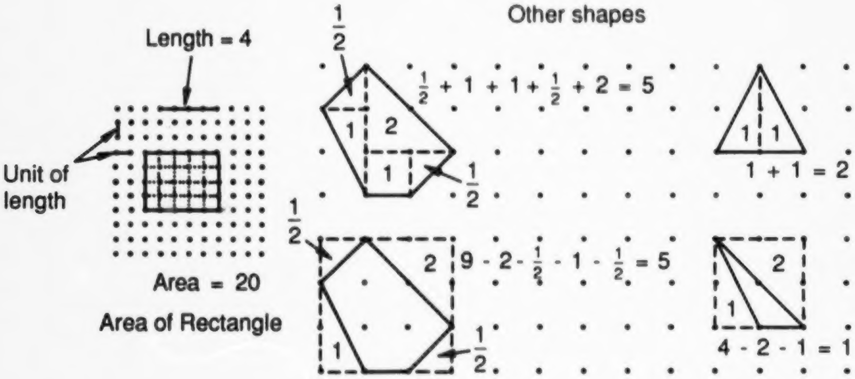
- Review (or introduce) the concepts of *interior angle*, *exterior angle*, and *supplementary angle*; have students find the exterior angles and the supplement of the interior angles of several triangles. Next, have them draw acute triangles using Logo graphics. Later, have them draw right and obtuse triangles. Have them record the exterior angles they use to make their triangles, find the associated interior angles, and see if the sum of the interior angles is 180 degrees.

- Using a large demonstration model or overhead transparency, discuss isosceles triangles. Present the definition in terms of the equal sides. Give students a worksheet containing several isosceles triangles and two nonisosceles triangles. The triangles should be placed in various orientations on the worksheet so that it is not immediately obvious which are isosceles. Have students measure the sides and angles of the triangles. See if you can lead them to make a generalization about the angles in an isosceles triangle.

- Ask students to make isosceles triangles using Logo. Discuss both the interior and exterior angles of isosceles triangles. Repeat the above activities for equilateral and right triangles.

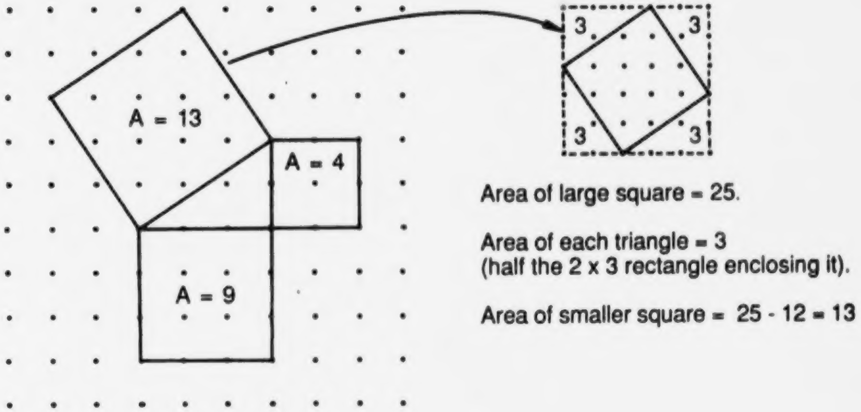
- Have students make acute, obtuse, right, equilateral, and isosceles triangles on geoboards. Teach them how to find the lengths on the geoboard. The distance between two adjacent vertical or horizontal pegs is defined as the unit of length on the geoboard. (See Figure 16.8.) Have them find the lengths of horizontal and vertical sides of triangles on the geoboard.

FIGURE 16.8
Areas on a geoboard.



■ Teach students how to find areas of shapes on the geoboard, as illustrated in Figure 16.8. Next, ask them to find the areas of squares built on the three sides of a right angle triangle on the geoboard (as illustrated in Figure 16.9), and record the results in a chart (as in the worksheet in Figure 16.10). Ask them to find and state the relationship between these three squares and they will be discovering for themselves the Pythagorean Theorem.

FIGURE 16.9
Using the geoboard to find the areas of squares.

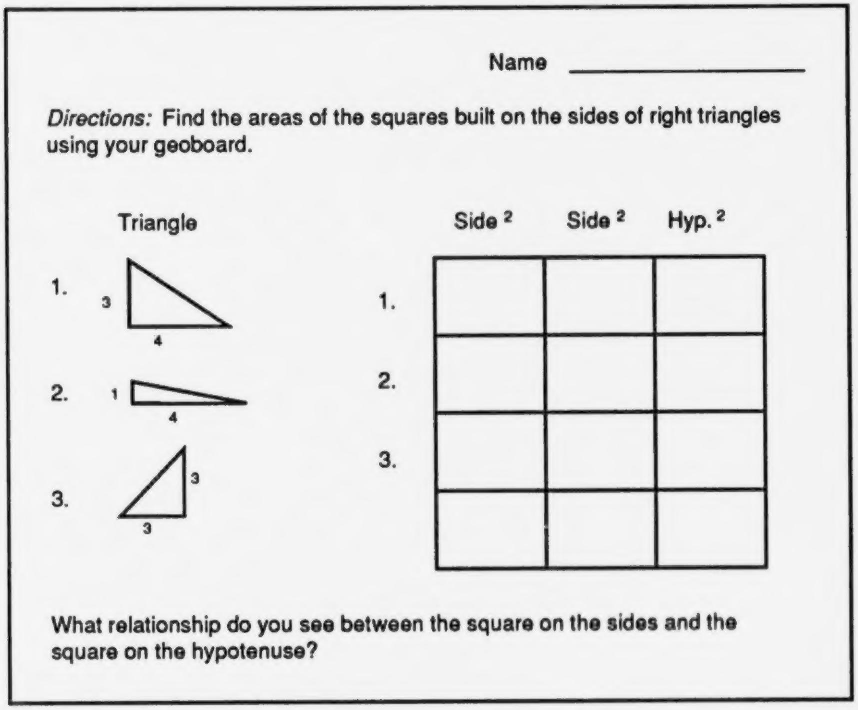


Area of large square = 25.

Area of each triangle = 3
(half the 2 x 3 rectangle enclosing it).

Area of smaller square = 25 - 12 = 13

FIGURE 16.10
A Pythagorean Theorem worksheet.



Name ____________________

Directions: Find the areas of the squares built on the sides of right triangles using your geoboard.

Triangle

1.

2.

3.

	Side ^{2}	Side ^{2}	Hyp. ^{2}
1.			
2.			
3.			

What relationship do you see between the square on the sides and the square on the hypotenuse?

- Discuss the Pythagorean rule with the students and then present it in algebraic form.

Side squared + Side squared = Hypotenuse Squared
or $A^2 + B^2 = C^2$

Once we have found the area of the square on the hypotenuse, we can find the *length* of the hypotenuse. It is the *square root* of the area (remember that the area of a square is found by squaring the length of the side).

C = The square root of A^2 plus B^2
or $C = \sqrt{A^2 + B^2}$

- Discuss the use of the Pythagorean theorem for finding the lengths of hypotenuses of right triangles. Give the students worksheets with right triangles whose two sides adjacent to the right angles are in exact centimeters (e.g., 4 cm and 3 cm). Ask them to use a centimeter ruler, the formula, and the square root key on their calculators or a square root table to find the length of hypotenuse. Next have them measure the hypotenuse and compare their measurement to the value obtained with the formula.

- Have them find the distance between diagonal pegs on the geoboard using the Pythagorean theorem.

- Have students write Logo or BASIC programs to calculate the length of the hypotenuse, given the length of the other two sides of a right triangle. Two possible programs are shown below.

BASIC

```
10 PRINT "TYPE THE LENGTH OF THE TWO SIDES"
20 PRINT "SEPARATED BY A COMMA AND HIT <RETURN>"
30 INPUT A,B
40 PRINT "HYPOTENUSE = "; SQR(A*A + B*B)
50 END
```

LOGO

```
TO RIGHTTR
  PRINT [TYPE THE LENGTH OF THE TWO SIDES]
  PRINT [NEXT TO THE RIGHT ANGLE IN A TRIANGLE]
  MAKE "X FIRST READLIST
  MAKE "Y FIRST READLIST
  PRINT (SE [THE HYPOTENUSE IS] SQRT(:X*:X+:Y*:Y))
END
```

Activities for Planes (Objective M)

As with abstract notions such as points and lines, the idea of a *plane* needs to be developed using models. A good model for a plane (a shape infinite in length and width but having no thickness) would be a flat piece of rubber; for example, a square cut from a large balloon. Have children hold each side and stretch it as far as they can. Have them imagine the balloon if it were stretched as big as the room, as big as the playground, and so on.

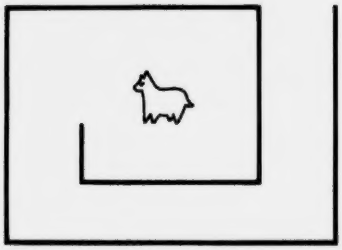
- Search for planes around the room.

- Discuss the figure formed when two planes meet (a line). Look for lines in the classroom formed by the intersection of two planes. What figure may be formed when three planes intersect?

Activities for Regions (Objective N)

Present the concept of *region* by explaining that a region is an area that is completely enclosed by a connected line, a curve or a combination of these. To help reinforce this concept, present "pens," some of which have "gates" or openings, others of which are completely enclosed. Explain that a region has no gates or openings that would allow an animal to get out. Show students several regions and nonregions on the chalkboard or overhead (see Figure 16.11) and use a stuffed animal (chalkboard) or transparency animal (overhead) to demonstrate the difference between regions and nonregions.

FIGURE 16.11
A region demonstration.

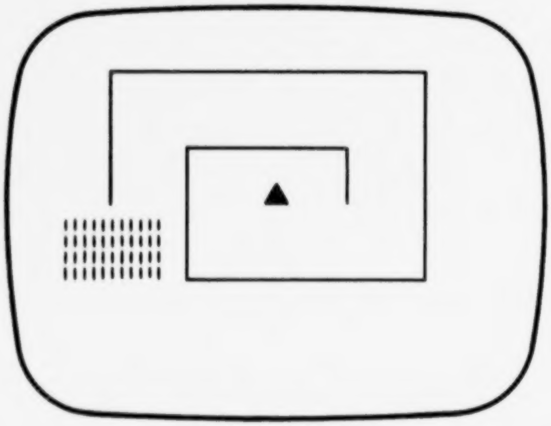


- Give younger children toy animals that they place inside the figures drawn on worksheets. Ask them to see if the animals can get out of the pens. Have them write "R" next to those that are closed.

- Execute the following procedure on the students' computers, using the Turtle Graphics in Logo. Tell them that the turtle is hungry and wants to get to the grass outside the pen, but cannot go through the fence to get to the grass. Ask the students to get the turtle out of the turtle pen to the grass without going through any fences. Have Logo beginners use the immediate mode. If they have been introduced to procedures, ask them to write a procedure to get the turtle to the grass.

```
TO TURTLEPEN
  MAKE "Y (-40)
  HT REPEAT 4 [MAKE "X (-120) MAKE "Y :Y - 10 REPEAT 10
   [PU SETPOS (SE:X :Y) PD FD 6 MAKE "X :X+8]]
  PU SETPOS [-100 0]
  PD FD 100 RT 90
  FD 160 RT 90 FD 140
  RT 90 FD 120
  REPEAT 2 [RT 90 FD 80]
  RT 90 FD 40 PU
  HOME ST PD
END
```

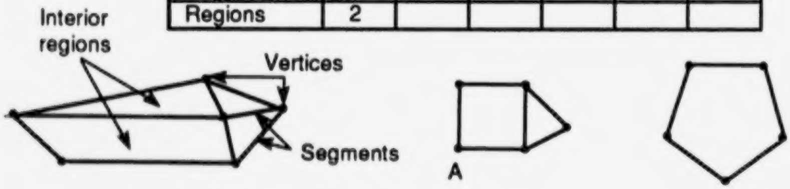
(no carriage return)



■ For older students, present the concepts of vertices, segments and regions. Give them an activity sheet, as in Figure 16.12, where they count the number of vertices, segments and regions in networks. After they have recorded the number of each, have them attempt to state the relationship between the number of segments, vertices, and regions in a network.

FIGURE 16.12
A network activity.

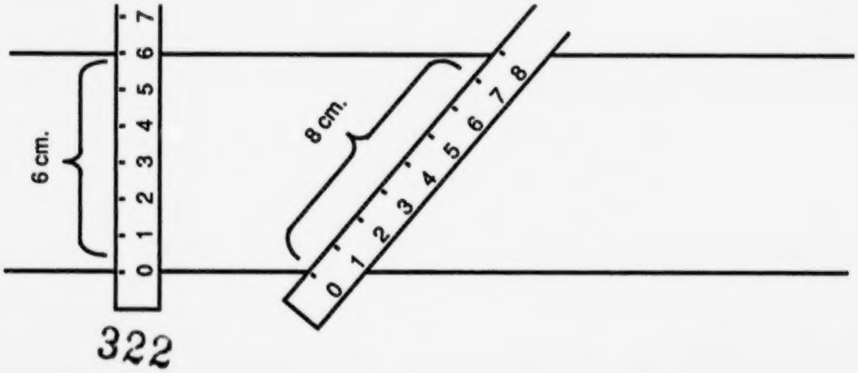
	A	B	C	D	E	F
Vertices	5					
Segments	6					
Regions	2					



Activities for Parallel Lines (Objective P)

Present the concept of *parallel*. Draw lines on the chalkboard or on an overhead transparency. Use a ruler to demonstrate that they are the same distance apart at all points along the lines. Try to lead students to discover that the way the ruler or other measuring device is placed is important when determining parallelism. (See Figure 16.13.) When is the ruler correctly placed?

FIGURE 16.13
Checking parallel lines.



- Have students make parallel lines on the geoboard, then ask them to "prove" that they are parallel (one argument would involve the length of the segments perpendicular to the parallel lines).

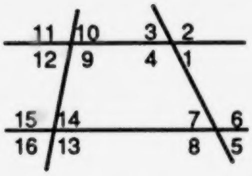
- Ask them to identify parallel lines in the classroom.

- Ask the students to draw parallel lines in Logo or BASIC graphics.

- Present and discuss intersecting lines. Have students use protractors to measure the four angles made when lines intersect. See if they can make a generalization about these angles.

- Repeat for a pair of parallel lines and a line intersecting them. Have students complete a chart (see Figure 16.14) with the measure of the eight angles formed by the line intersecting the pair of parallel lines, then ask them to generalize about the angles formed.

FIGURE 16.14
A parallel and intersecting line activity.



Directions: Write the amount of each angle.

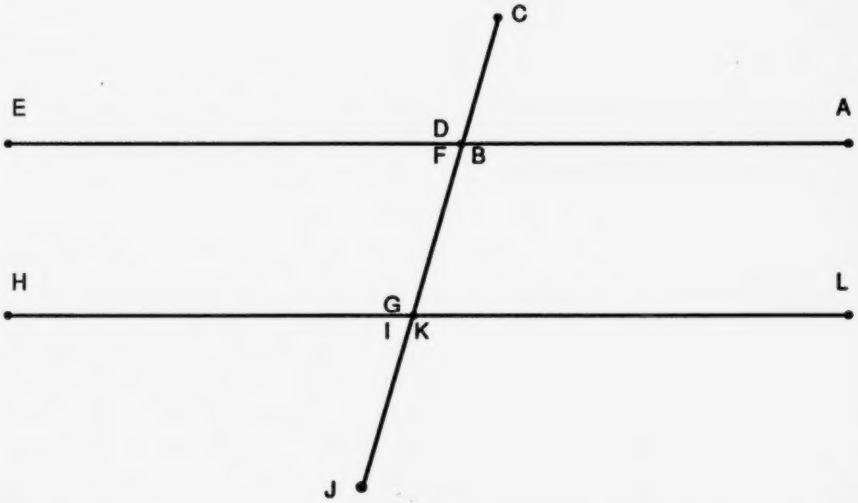
1	3	5	7	9	11	13	15
2	4	6	8	10	12	14	16

And so on.

- In order to help reinforce the above concepts, ask students to write a Logo procedure that creates a pair of parallel lines and an intersecting line. Direct students to draw with their program the parts of the figure in the order indicated by the letter sequence of the points. (See Figure 16.15.)

FIGURE 16.15
A Logo activity for a line intersecting parallel lines.

Directions: Write a logo procedure that starts at A, goes to B, C, D, etc. Segment AE, LH, and CJ should be straight and AE should be parallel to LH.

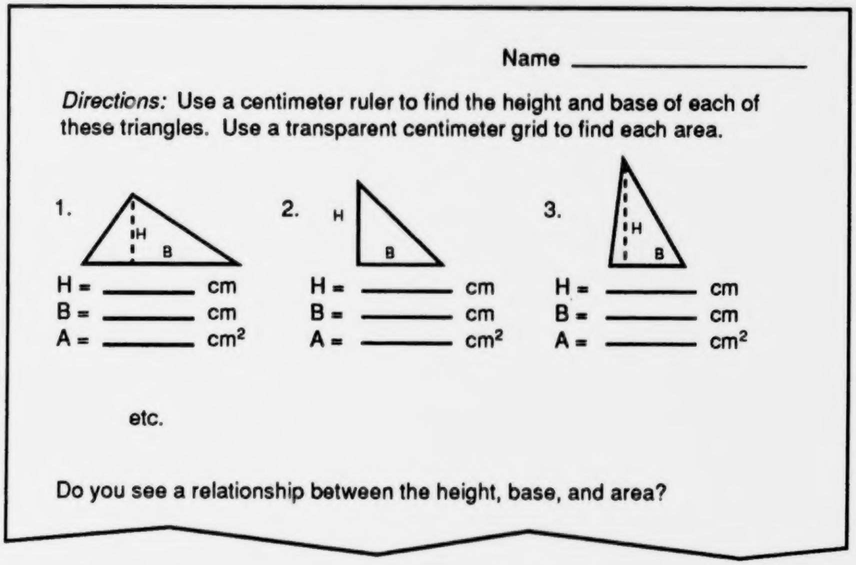


Activities for Rectangles (Objectives Q–R)

After students have been given the basic concepts of geometry as they relate to triangles and have applied the concepts in activities like those discussed above, it is not difficult to extend their knowledge to squares, rectangles, and other polygons. Start by defining *square* (a shape with opposite parallel sides, adjacent equal sides, and a right angle), presenting examples of squares, and discussing their attributes.

- Ask students to construct squares using straight edges and right angles cut from tagboard.
- Have them draw a square using Logo.
- Have students find examples of squares in the classroom and around the school.
- Define, present examples, and discuss *rectangle* (a shape with opposite parallel sides and a right angle); then do activities similar to those presented for squares.
- Have students find areas and perimeters of triangles and squares using the *grid* technique introduced in Chapter 15.
- Introduce students to the concept of *height* of a triangle. Give them a worksheet, such as the one in Figure 16.16, with a series of triangles. Have them find and record the lengths of the base and the heights using a centimeter ruler, and the areas using a centimeter grid (see Chapter 15). Ask them to generalize about the relationship between the length of the base, the height, and the area of triangles.

FIGURE 16.16
A worksheet for discovering the formula for the area of triangles.



Name ____________________

Directions: Use a centimeter ruler to find the height and base of each of these triangles. Use a transparent centimeter grid to find each area.

1.
H = ________ cm
B = ________ cm
A = ________ cm^2

2.
H = ________ cm
B = ________ cm
A = ________ cm^2

3.
H = ________ cm
B = ________ cm
A = ________ cm^2

etc.

Do you see a relationship between the height, base, and area?

- Do similar activities with rectangles: have students measure the length, width and area of rectangles, record them in a chart, and attempt to generalize about the relationship between these values. Of course, after the activity has been completed, the teacher should discuss the results, including the formula, and give students practice activities.

■ Make a set of 5 to 8 rectangles all having the same perimeter (e.g., 4 by 6, 5 by 5, 3 by 7, 2 by 8, 1 by 9, etc.). Have the students measure and record the length, width and area of each and make a graph with the length on one axis and the area on the other. Ask them to generalize about the relationship between the lengths of sides of rectangles with constant perimeters, and the rectangles' area. Which length-width combination produces the greatest area?

Activities for Polygons (Objectives S–T)

Introduce the term *polygon* (including the root meanings of *poly* and *gon*) and the prefixes used in identifying polygons.

Penta—five
Hexa—six
Septa—seven
Octa—eight
Nona—nine
Deca—ten

Present students with a variety of *irregular polygons* and have them identify them according to name.

Introduce the concept of *regular polygon* (a polygon with equal sides and angles) and present examples.

■ Have students draw a regular pentagon using Logo. First, have them draw each side explicitly. Next, have them use the REPEAT command to draw the polygon. Repeat for other regular polygons.

■ For advanced and upper-level students, ask them to write a Logo procedure that will accept a number from the keyboard and then draw a regular polygon with that many sides.

■ Introduce the definitions for *trapezoid, parallelogram,* and *rhombus,* in terms of their sides. Have students investigate the relationship between the angles of parallelograms and rhombuses.

■ After appropriate introductions, ask the students to find the areas of various trapezoids, parallelograms, and rhombuses.

Activities for Solid Geometric Shapes (Objectives U–V)

At an early age, children should be introduced to the difference between solid geometric shapes (like spheres) and their two-dimensional counterpart (circles). With kindergarteners or first graders, exact terminology is not necessary (e.g., rectangular solid); common terminology will do. The following is a list of solid geometric shapes that children may be introduced to and the beginning vocabulary that we can use.

Solid Shape	*Early Terminology*
Sphere	Ball
Cube	Cube
Rectangular Solid	Box
Cone	Cone
Cylinder	Can or Cylinder
Tetrahedron	Pyramid
Other Pyramids	Pyramid

- Introduce the different solid figures using commercial wooden or teacher-made models. Discuss the differences between the different figures.

- Look for examples of the different solid shapes around the school. Record the shapes on paper or in a data-base program (see activity above for two-dimensional geometric shapes).

- Place cardboard boxes in an appropriate place in the classroom, each labeled with the name of a type of solid figure: cylinder, rectangular solid, cone, and so forth. Have children bring to class examples of the different types of solids. When a child comes with a new object have her share it with the class and ask the class to decide which box it belongs in. Put the child's name on the object before placing it in the box so that it may be reclaimed after the activity is completed.

- For middle and upper-grade students, after they have mastered measuring the areas of plane figures, introduce them to methods for measuring the surface area of solid figures. To introduce the concept of measuring the surface area of solid figures, use centimeter grid paper (as discussed in Chapter 15) to find the surface area of cylinders. Have students trace all sides of a cylinder on the grid paper, then count the number of square centimeters contained in the regions that have been drawn. (See Figure 16.17.) Repeat for a second cylinder. Note that if you cut out the top and bottom of a cylinder with a can-opener and cut the side with a tin-snip and lay it flat, the resulting shapes are two circles and a rectangle. Gifted upper-grade students may be able to develop a formula for finding areas of cylinders after such an activity.

- For other shapes such as rectangular solids and pyramids, discuss the shape of the faces, then use formulas for finding the surface areas of several of the shapes.

- Introduce the formulas for finding the volume of rectangular solids (including cubes). Have the students calculate the volume of several figures in cubic centimeters, then find the volume of the same shapes using water displacement. Compare the volumes obtained using the two techniques.

Activities for Congruency, Similarity, and Symmetry (Objectives W--Y)

For young children, *congruency* (figures having the same size and shape) may be demonstrated by placing one shape on the other. If all points on the boundary of the two shapes match, they are congruent.

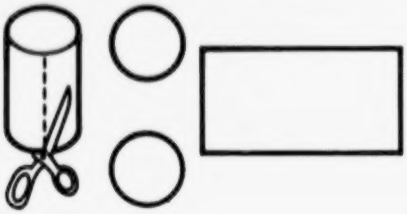
- Make a variety of shapes—squares, circles, triangles, parallelograms, and irregular figures—out of colored construction paper. Give a set of 20 or so of these shapes to groups of 3 or 4 children and ask them to find those pairs of shapes that are congruent.

- Make a card game with 20 pairs of congruent shapes in random colors (the color should not clue children as to whether the shapes are congruent). Include a card with a black star. The rules of play to use are the same as in "Kiss the Spider," introduced in Chapter 20.

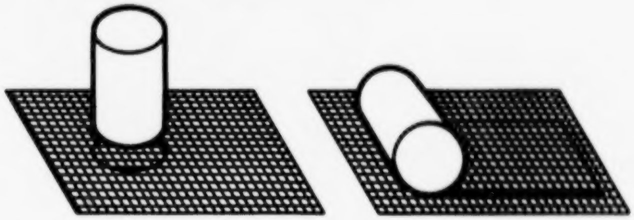
Introduce the concept of *symmetry* on a flannel board, chalk board, or overhead. First, discuss lines of symmetry in a single shape. A shape is symmetric when

FIGURE 16.17
Finding the surface areas of cylinders.

1. Trace around the top and bottom of the can on grid paper.



2. Make a mark on the side of the cylinder. Place the mark so that it touches the grid and roll it along the grid until the mark touches the grid again. Draw a line enclosing all the area that touched the grid while you were rolling the cylinder. It should be in the form of a rectangle. Record the resulting areas in the chart below.



	Top Area	Bottom Area	Side Area	Total Area
Tin Can #1				
Tin Can #2				

a line may be placed through the figure so that it splits the shape into two mirror images. Use mirrors or Miras® to illustrate this concept (see Figure 16.18). A Mira is a device made of red plastic that shows the mirror image of an object and at the same time is transparent, allowing the user to see the figure on the other side of the device. This allows the user to "map" one shape on to another, or part of a shape on to another part of the same shape. Using this device, it is a simple task for a student to find the line of symmetry of a shape, to find the perpendicular bisector of a line, or to do transformations.

■ Give students worksheets containing circles, squares, rectangles, and other figures and ask them to draw the line(s) of symmetry for these figures. If Miras are available, use them for this activity. If not, have children fold cutouts of the figures on the worksheets to match halves of the figures (see Figure 16.19).

■ Do activities similar to those above for pairs of shapes. Discuss which types of figures always have lines of symmetry, no matter how they are placed on paper, and which do not.

FIGURE 16.18
Using a Mira to find a line of symmetry.

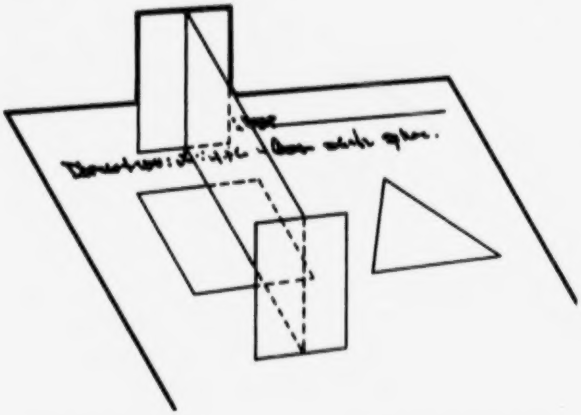
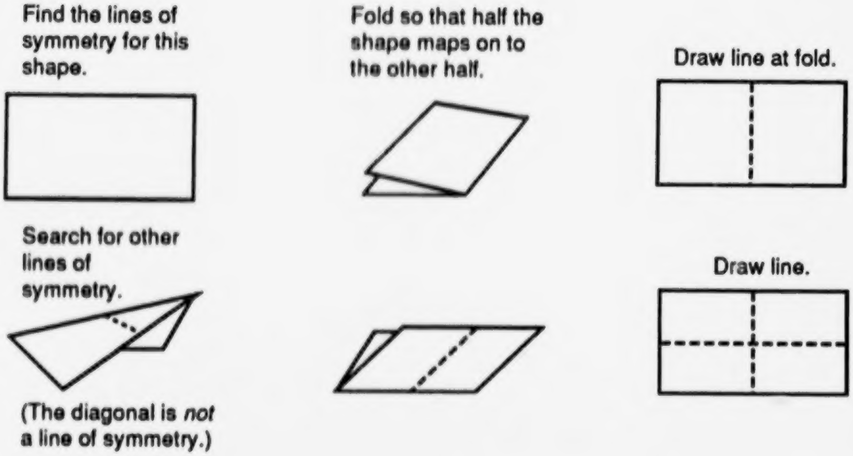

Directions:

FIGURE 16.19
Using paper folding to find lines of symmetry.


Find the lines of symmetry for this shape.
Fold so that half the shape maps on to the other half.
Draw line at fold.
Search for other lines of symmetry.
Draw line.
(The diagonal is *not* a line of symmetry.)

Activities Involving Symmetry and Similarity for Older Students

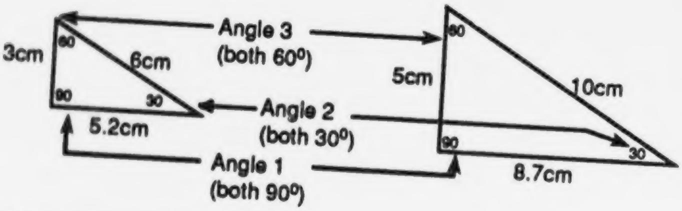
After students understand the concept of *proportion* (Chapter 14), introduce the definition of *similarity* (figures that have the same shape but not necessarily the same size). Similar figures have *congruent* angles and their corresponding sides are *proportional.* To check to see if polygons are similar, measure all angles of both figures; if all corresponding angles are congruent the figures are similar. Fig. 16.20 shows pairs of similar triangles and non-similar quadrilaterals.

- Give students a variety of pairs of shapes and have them use angle checkers or protractors to see which pairs are similar.

- With triangles it is possible to determine similarity by checking to see if all corresponding sides are proportional. (Since triangles are rigid figures, their angles are determined by the lengths of their sides—you can make only one triangle with sides of 3 cm, 4 cm and 5 cm.) One way to check the proportionality of the sides of two triangles is with the calculator: divide one side of one triangle by its corresponding side in the other figure; then, multiply each side of the first triangle by this factor and check to see if the value obtained is equal to the corresponding side in the other figure. See the example in Figure 16.21.

FIGURE 16.20
Similar figures.

Example 1:

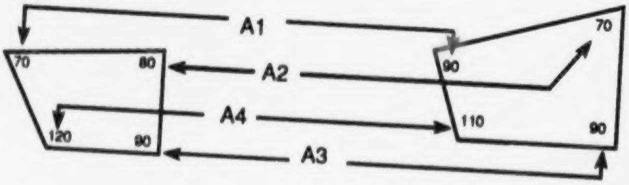

3cm
6cm
60
90
30
5.2cm
Angle 3
(both 60°)
Angle 2
(both 30°)
Angle 1
(both 90°)
5cm
10cm
60
90
30
8.7cm

They are similar since corresponding angles are proportional.

Note that $\frac{3\text{ cm}}{5\text{ cm}} = \frac{6\text{ cm}}{10\text{ cm}} = \frac{5.2\text{ cm}}{8.7\text{ cm}}$

lengths of corresponding sides

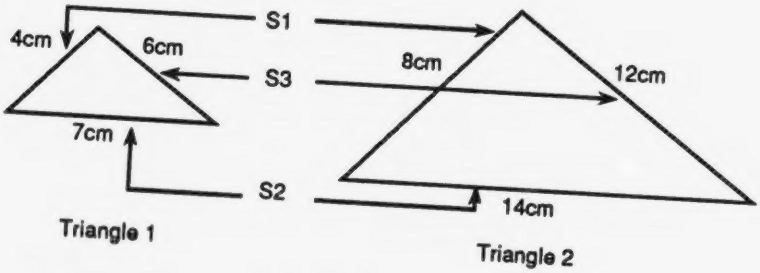
Example 2:


A1
A2
A4
A3
70
80
120
90
70
90
110
90

Angle 1 70° ≠ 90°
Angle 2 80° ≠ 70°
Angle 3 90° = 90°
Angle 4 120° ≠ 110°

They are not similar.

FIGURE 16.21
Using a calculator to determine similarity of triangles.


S1
S3
S2
4cm
6cm
7cm
8cm
12cm
14cm
Triangle 1
Triangle 2

1. Relationship between sides 1 $8\,\boxed{\div}\,4\,\boxed{=}$ (2)
2. Multiply this factor by other sides of triangle 1 to see if you get corresponding sides in triangle 2.

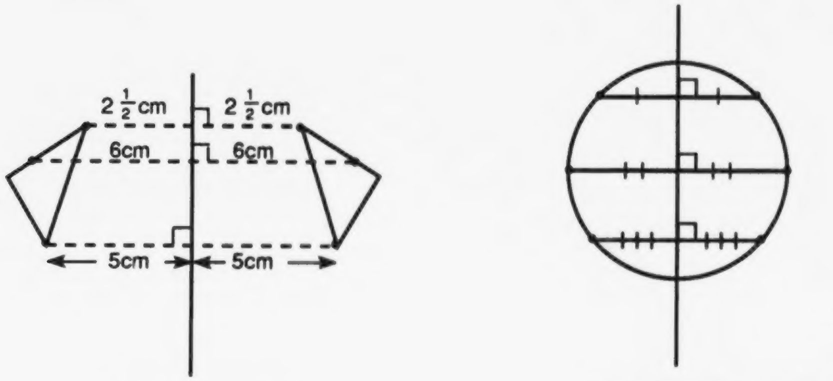
$$\left.\begin{array}{l} 7\,\boxed{\times}\,2\,\boxed{=}\ (14) \\ 6\,\boxed{\times}\,2\,\boxed{=}\ (12) \end{array}\right\}$$ Yes, the sides are proportional.

- Have students use Logo to draw similar figures. Ask more advanced students to write a procedure that asks the user to type a factor, then draws two equilateral triangles where the length of the sides of the second is multiplied by the factor. For example, if the factor is 2, the sides of the second triangle are twice as long as those of the first triangle. Students who complete this assignment quickly could be asked to modify the procedure so that when the user types the number of sides as well as the factor, the computer draws similar regular polygons with the number of sides indicated, and with the sides of the second multiplied by the given factor.

- Present different, similar, and nonsimilar shapes. Have students determine if the corresponding angles of the shapes are *equal* in measure. Next, have them measure the sides of the figures and determine if they are in *proportion*. Finally, have them determine if the shapes are *similar.*

- Discuss the definition of *line symmetry* (a line through a shape or between two shapes that is the perpendicular bisector of all line segments connecting corresponding points on the two figures or on the two halves of the figure). See Figure 16.22.

FIGURE 16.22
Lines of symmetry.



Line of symmetry of two shapes

Line of symmetry of one shape

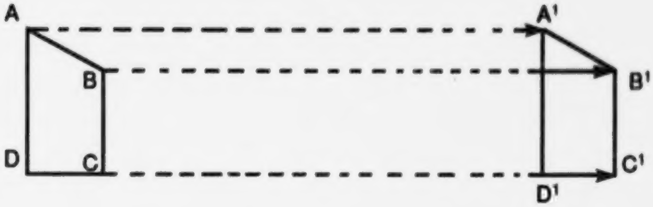
(I, II, or III indicates congruent segments)

- Have students use Miras or protractors and rulers to determine if shapes on paper are symmetric with respect to a given line. Also, give them pairs of congruent shapes, which may or may not have lines of symmetry, and ask them to find and draw the line of symmetry. Note that when you use a Mira, two shapes or a single figure have a line symmetry if one figure can be mapped on the other figure: that is, if the Mira can be placed so that the reflection of one shape corresponds with the other shape. With the Mira it is easy to discuss the idea of *reflection*—each point of one shape is the *image* of the corresponding point in the other figure under reflection.

Another way of approaching the idea of line symmetry is through *transformations.* Rather than the static comparisons of shapes to determine symmetry, imagine flipping one of the shapes over the line (keeping all points of the shape being flipped at the same distance from the line at all times) and matching it to the other shape. The easiest way to illustrate this is to use paper folding as discussed above. If the two shapes match perfectly, one is said to be the *flip image* of the other.

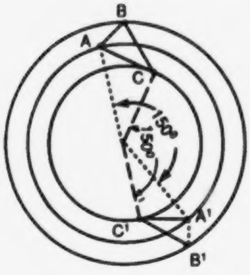
Another type of transformation is the *translation* or slide. Imagine a tagboard trapezoid on a sheet of paper (Figure 16.23). Draw around the shape, then slide it along the paper so that each vertex moves exactly the same distance (i.e., *do not* rotate the shape—the imaginary lines made by the movement of all vertices are parallel). Now, draw around the trapezoid at the new position. One figure is the image of the other through translation.

FIGURE 16.23
Translations.



A third type of transformation is a *rotation* or turn. Here, two or more shapes are symmetric with respect to a point. (See Figure 16.24.) Imagine moving a triangle so that each point on the figure moves on a circle and all circles created by the movement of the points have the same center (in other words, they are *concentric*). Note that the angles made by connecting each point and its image (i.e., the corresponding point in the new position) with the center of the circles are congruent.

FIGURE 16.24
Rotations.



■ After appropriate introduction and discussion, have students use Miras, compasses, straightedges, and protractors to determine if figures are the images of other figures through flipping, translation, or rotation. Also, give them figures and have them construct shapes that are their images through various translations.

Sometimes teachers feel that the meat of elementary mathematics relates to number concepts and operations while geometry is somewhat esoteric and unimportant. To the contrary: geometry is an important component of the mathematics curriculum and should not be ignored. Not only is knowledge of geometry useful in everyday life, it is fundamental to much of the mathematics and science that students encounter later. In a technological society, mathematics involves more more than adding, subtracting, multiplying, and dividing.

SUGGESTED READINGS

Arithmetic Teacher 26 (6), February 1979. This issue deals with teaching geometry.

Campbell, P.F. Cardboard, rubber bands, and polyhedron models. *Arithmetic Teacher* 31 (2), October 1983: 48–52.

Dana, M.E., and Linquist, M.M. Let's try triangles. *Arithmetic Teacher* 26 (1), September 1978: 2–9.

______. The surprising circle. *Arithmetic Teacher,* 25 (4), January 1978: 4–10.

Hawkins, V.J. The pythagorean theorem revisited: Weighing the results. *Arithmetic Teacher* 32 (4), December 1984: 36–37.

Holcomb, J. Using geoboards in the primary grades. *Arithmetic Teacher* 27 (8), April 1980: 22-25.

Lappan, G., and Winter, M.J. Spatial visualization. *Mathematics for the Middle Grades* (1982 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1982.

Mira Math for Elementary School. Mira Mathematics Corporation.

Mira Activities for Junior High School Geometry. Mira Mathematics Corporation.

Nelson, G., and Leutzinger, L.P. Let's take a geometry walk. *Arithmetic Teacher* 27 (3), November 1979: 2-4

Ott, J.M., Sommers, D.D., and Creamer, K. But why does C = d? *Arithmetic Teacher* 31 (3), November 1983: 38-41.

Van de Walle, J. and Thompson, C.S. Concepts, art, and fun from simple tiling patterns. *Arithmetic Teacher* 28 (3), November 1980: 4-8.

______. A triangle treasure. *Arithmetic Teacher* 28 (6), February 1981: 6-11.

Wenninger, M.J. *Polyhedron Models for the Classroom.* Reston, VA: Virginia: National Council of Teachers of Mathematics, 1966.

Zurstadt, B.K. Tessellations and the art of M.C. Escher. *Arithmetic Teacher* 31 (5), January 1984: 54-55.

QUESTIONS

1. Write a Logo procedure to draw a triangle that looks like the one in Figure 16.25. End with the turtle at the top of the triangle. Now, determine a method for drawing a line segment from the top vertex to the base of the triangle that is exactly the altitude of the triangle—it must be perpendicular to the base and end exactly on the triangle's base. Hide the turtle as you draw the height so that you can tell if the line is the right length.

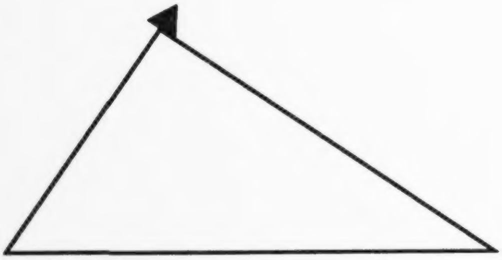


FIGURE 16.25
Logo triangle.

2. Develop a device for the classroom that children may use to determine if two lines are parallel. (Note: The "ruler" you develop should always be in the correct position to measure the distance between the lines.)

3. Write a Logo program that draws two parallel lines and an intersecting line without doing a PENUP command. Follow the sequence of points in Figure 16.15—have the turtle go from point A to B, turn and go to C, turn around and face in the opposite direction, and go to D, and so on.

4. Describe an activity to help children develop a formula for finding the surface area of a rectangular solid, such as a wooden block 10" long, 2" wide, and 4" deep.
5. Write a Logo procedure that will accept a number from the keyboard and then draw a regular polygon with that many sides.
6. Find a rule that gives the total number of degrees in all the angles in a regular polygon with *n* sides.
7. Discuss how the geoboard could be used to teach similarity of figures.

CHAPTER

17

Integers, Real Numbers, & Other Topics

The statement, "numbers are everywhere," is true for a variety of types of numbers. It is certainly true for whole numbers, decimals, and percentages. It is true to a lesser degree for integers and nonrational real numbers. If you live in the northern part of the North American continent, you need to know the difference between – 15° Celsius and 15° Celsius if you want to use the weather forecast to help you select the proper attire on a winter day.

If you have a checking account and you get a note from the bank stating that your balance is – $85.38, an understanding of negative rational numbers would help you realize that you should rush to the bank with some money rather than writing a $50.00 check at a restaurant.

If you are an engineer or a carpenter, you might need to determine the exact length of the diagonal of a rectangle based upon the sides of the rectangle—this involves real numbers (square roots).

TEACHING ABOUT NUMBERS

Teaching children about the kinds of numbers discussed above is important because it prepares them for real-world experiences with mathematics. It also prepares them for the more advanced study of mathematics and the rise of technology. In this chapter we consider integers, real numbers, and some additional topics related to numbers.

The *set of integers* is made up of the counting numbers, the negatives of these numbers, and zero (positive and negative whole numbers). The ellipses in Figure 17.1 indicate that the set of integers is infinite in both the positive and negative di-

FIGURE 17.1
The set of integers.

{. . . -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 . . .}

rections. When teaching students about the set of integers, we need to present the concept of negative numbers and operations on positive and negative numbers through concrete examples and models. We also need to convey the idea of an infinite set.

The *set of real numbers* contains the set of integers (which contains the set of whole numbers), the set of rationals (positive and negative), and the set of all nonrationals. As you can see in the diagram in Figure 17.2, the sets of whole numbers, integers, and rational numbers are subsets of (i.e., completely contained inside) the *set of reals* or real numbers.

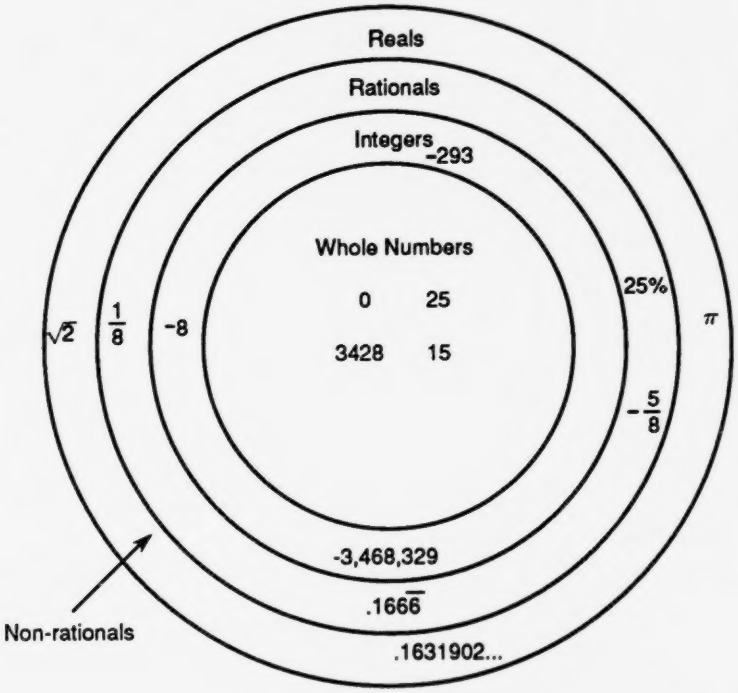


FIGURE 17.2
The set of real numbers

As is mentioned in Chapter 12, the *set of rational numbers* contains all numbers that can be expressed as a fraction (a/b) or as a repeating decimal (e.g., 1.3333333. . . or −.142857142857142857. . .). The *set of nonrational reals*, then, is the set of all numbers that cannot be expressed as fractions or repeating decimals (e.g., 23.18245192 . . .). Examples of nonrational reals are *pi* (3.14159. . .) and $\sqrt{2}$ (1.41421356. . .).

The principles like the commutative property and the associative property discussed earlier apply to integers and to real numbers:

- The commutative properties for addition and multiplication:

$$^-3 + {}^-6 = {}^-6 + {}^-3$$
$$^+5 \times {}^-8 = {}^-8 \times {}^+5$$

- The associative properties for addition and multiplication:

$$({}^{+}23 + {}^{-}17) + {}^{-}5 = {}^{+}23 + ({}^{-}17 + {}^{-}5)$$
$$({}^{+}25 \times {}^{-}36) \times {}^{+}4 = {}^{+}25 \times ({}^{-}36 \times {}^{+}4)$$

- The distributive property:

$${}^{-}3 \times (5 + {}^{-}.333) = ({}^{-}3 \times 5) + ({}^{-}3 \times {}^{-}.333)$$

- The *identity* for multiplication is 1 and the *zero* is 0. The *identity* for addition is 0.

Behavioral Objectives for the Real Number System:

A. Give two examples of integers in an applied setting.
B. Add, subtract, multiply, and divide integers.
C. Find the absolute value of an integer.
D. Identify prime numbers and find the prime factorization of composite numbers.
E. Give two examples of irrational real numbers in an applied setting.
F. Find the square root of a given number using estimation and checking.

Activities for Integers (Objectives A–C)

Although the idea of an integer is somewhat abstract, we have numerous everyday examples on which to build a foundation of understanding.

Negative Integers

■ If you live in a location where negative temperatures regularly occur, show students a thermometer and discuss what the weather is like at both positive and equivalent negative temperatures.

■ Another way to introduce negative integers is to discuss money. Have them use their calculators to find the answers as you discuss the following situation: *Suppose that Nancy has 95 cents; she goes to the store to buy a candy for 37 cents, including tax. How much does she have left? Next, she goes to the stationary store and buys a birthday card for her friend Nancy for 48 cents. Then she goes to the Post Office to buy a 20 cent stamp so that she can send the card. How much will she have to borrow from her friend Juanita so that she can buy the stamp? How much money does she have after buying the stamp?*

■ Introduce the number line containing integers in Figure 17.3 to your class. Discuss the line, its symmetry, the meaning of the arrows, and so forth. Have students locate positive and negative values on the line.

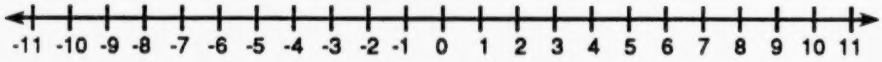
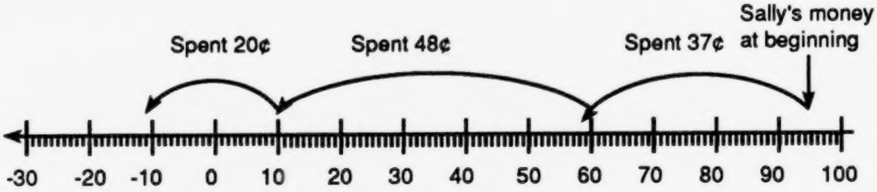


FIGURE 17.3
The integer number line.

Addition and Subtraction of Integers

■ Introduce the idea of operations on integers by continuing the discussion of Nancy and her money. Locate the amount of money Nancy has on the number line in Figure 17.4 and have students use their calculators to determine and verify results.

FIGURE 17.4
A problem using the integer number line.


Sally's money at beginning
Spent 20¢
Spent 48¢
Spent 37¢
-30
-20
-10
0
10
20
30
40
50
60
70
80
90
100

On Saturday, Nancy gets her 50-cent allowance, pays back Juanita, and deposits it in her piggy bank. Later that day, she finds a quarter (25 cents) that she also deposits. Discuss adding a positive amount and include an explanation of the signs indicating positive numbers. (See Figure 17.5.) *The next day, she spends 65 cents on a comic book, leaving her with nothing in her piggy bank.* (See Figure 17.6.) *On Monday, she needs to buy a pencil at the school store and borrows 20 cents from Juanita. When she gets home, she places an IOU in her piggy bank to remind herself that she owes Juanita money. That same day, she borrows another 10 cents from Juanita for candy and replaces the old IOU with another one. How much money does Nancy have?* (See Figure 17.7.)

FIGURE 17.5
Adding a positive.

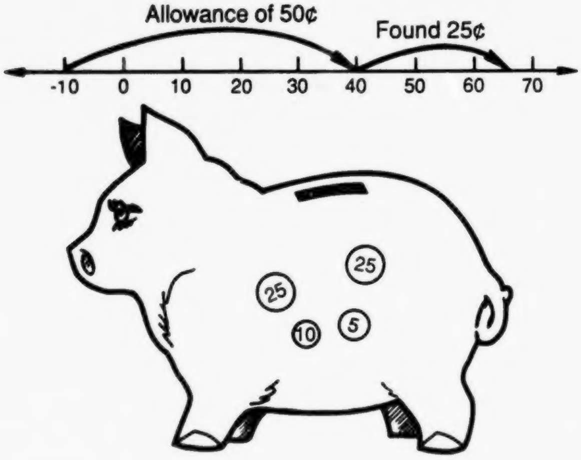

Allowance of 50¢
Found 25¢
-10
0
10
20
30
40
50
60
70
25
25
10
5

FIGURE 17.6
Subtracting a positive.

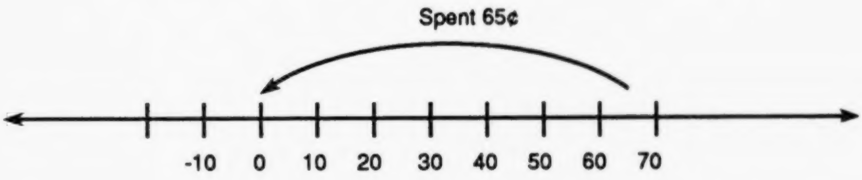
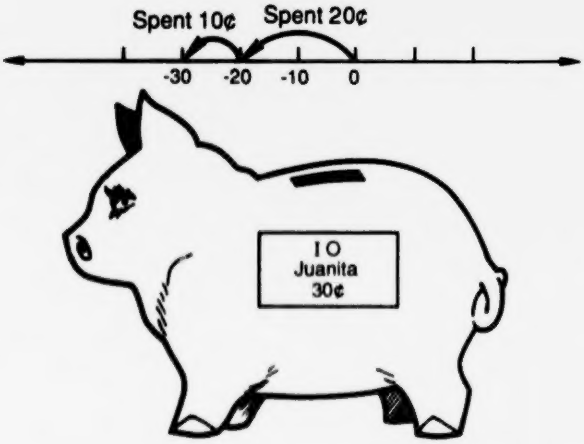

Spent 65¢
-10
0
10
20
30
40
50
60
70

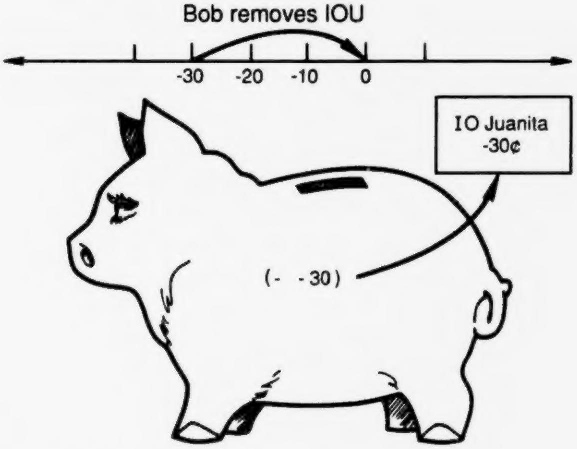
FIGURE 17.7
Subtracting a positive.



■ At this point, give students practice with verbal problems involving integers. Have them use number lines to solve them. For example: *Mr. Thomas has $50 in his bank account and writes a check for $75. How much money will he have after the check clears (i.e., is paid by the bank)?*

■ Next, the idea of subtracting a negative value can be introduced, as shown in Figure 17.8. *It is Nancy's birthday. Her brother, Bob, as a birthday present, offers to pay the IOU to Juanita and removes the IOU from her piggy bank* (subtracting a negative amount). At this point, discuss what happens when you subtract a negative. Students with help should now be able to summarize the operations on integers.

FIGURE 17.8
Subtracting a negative.



1. Adding a positive (e.g., putting money in the piggy bank) increases the amount in the bank (see Figure 17.9).
2. Taking money out decreases the value in the bank (see Figure 17.10).

FIGURE 17.9
Adding a positive.

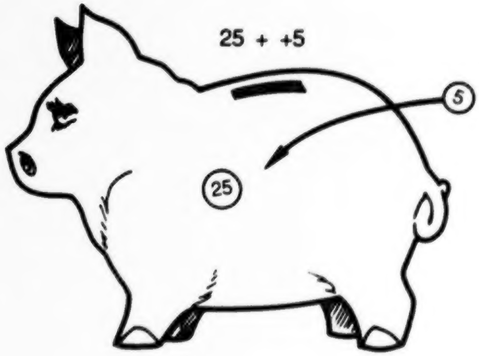
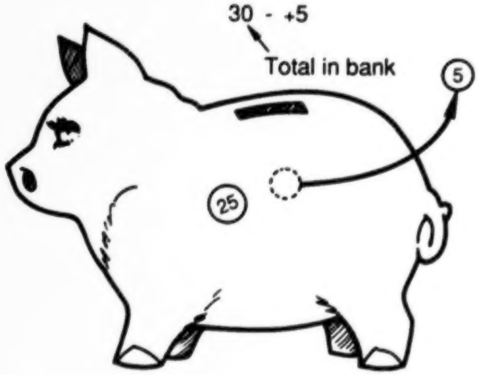
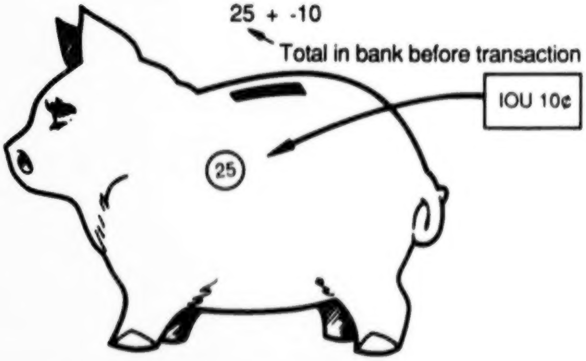


FIGURE 17.10
Subtracting a positive.



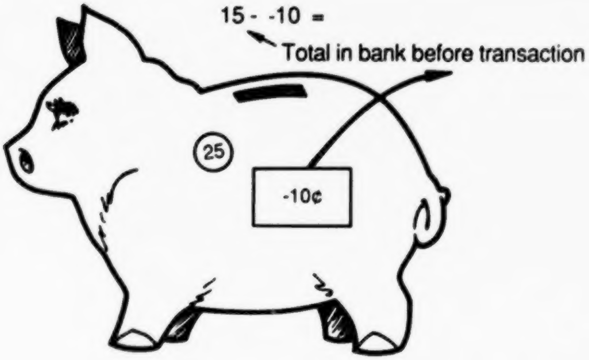
3. Placing a negative amount in the bank by depositing an IOU decreases that amount (see Figure 17.11). At this point, give examples where a child has an amount in the bank (e.g., 98 cents) and an IOU is placed in the bank (e.g., for 32 cents). How much does the piggy-bank owner actually have?
4. Removing a negative amount (removing an IOU) increases the value of the money in the bank (see Figure 17.12).

FIGURE 17.11
Adding a negative.


25 + -10
Total in bank before transaction
IOU 10¢
25

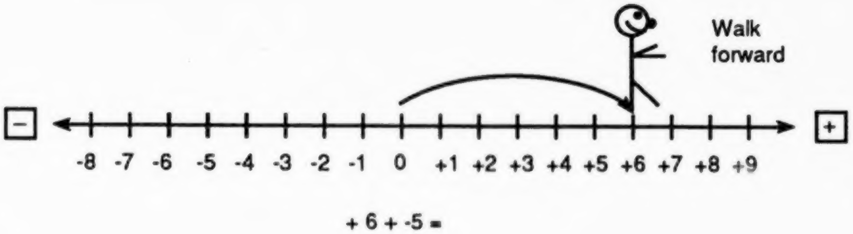
The idea of using number lines for integers is introduced above. Make a large number line using a roll of butcher paper with the integers about a foot apart. Place a large + sign on the wall on the positive side of the number line and a − sign

FIGURE 17.12
Subtracting a negative.


15 - -10 =
Total in bank before transaction
25
-10¢

on the opposite side. Have students walk to various points on the number line to indicate operations on integers, as illustrated in Figure 17.13. You may wish to place instructions indicating which way to face and walk on the chalkboard in the initial stages. The first step is to walk to the first number, whether positive or negative. Next, there are four possibilities (see Figures 17.14 to 17.17):

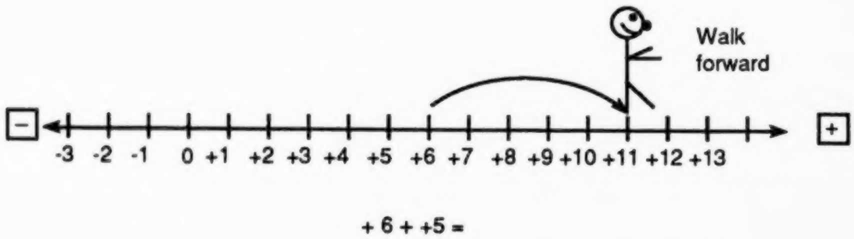
FIGURE 17.13
Walking on a number line.



1. Add a positive number—face in the positive direction, walk forward
2. Subtract a positive—face in the negative direction, walk forward
3. Add negative—face in the positive direction, walk backward
4. Subtract a negative—face in the negative direction, walk backward

When discussing the similarity between pairs of the operations above, avoid saying that "adding a negative is the same as subtracting a positive," or "subtracting a negative is the same as adding a positive." Even though the results of these pairs of operations are the same, the operations are quite different, as witnessed by the examples above. Of course, students will discover that you can find the answer

FIGURE 17.14
Adding a positive.



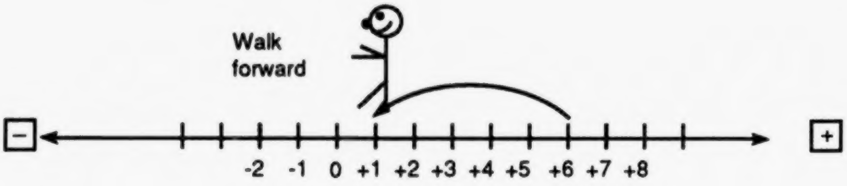


FIGURE 17.15
Subtracting a positive.

$+6 - {}^{+}5 =$

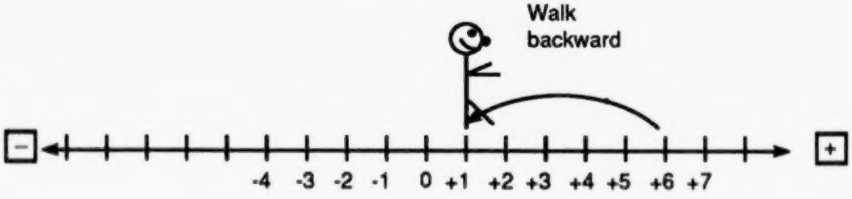


FIGURE 17.16
Adding a negative.

$+6 + {}^{-}5 =$

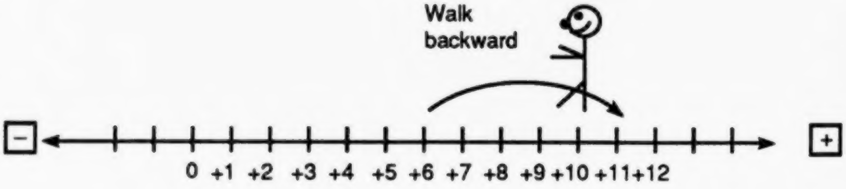


FIGURE 17.17
Subtracting a negative.

$+6 - {}^{-}5 =$

to a problem where a negative is subtracted by adding. It is important, however, for us to stress meaning so that students will be able to understand how and when to apply a given operation. Therefore, present the four operations separately and ask students to use and/or to picture number lines to help decide which operation to use.

■ After students have experienced operations on smaller integers, extend the activities to include larger or smaller (i.e., multidigit negative) numbers like the following:

$${}^{-}347 + {}^{-}492 =$$

■ For older students, include a few operations on negative fractions and decimals to point out that the principles learned relative to integers also apply to rational numbers:

$${}^{+}\frac{3}{4} - {}^{-}\frac{1}{2} =$$

$${}^{-}.75 - {}^{+}1.27 =$$

■ In order to have students use calculators to solve problems involving negatives, you will need a calculator that allows you to enter and display negative numbers. Such calculators usually have a +/− key that allows you to enter a value, then change its sign by pressing the key. It is not possible to add or subtract negative numbers on calculators without this provision. After introducing students to the technique of adding or subtracting negatives, have them check the results of their computations with the calculator.

■ Have students experiment with adding and subtracting negative numbers in BASIC or Logo. This may be done in the immediate mode and without any previous experience with computers. The following are four BASIC examples of how the print statement may be used to find answers to problems involving addition and subtraction of integers. Note that it might be helpful to enclose the integer and its sign in parentheses since it is not possible to print a superscripted sign as can be done on paper.

```
PRINT +5 - +3
PRINT -20 + (-15)
PRINT +10 - (-5) - (+12)
PRINT (-5) + (-9)
```

Multiplication and Division of Integers

Review the idea of multiplication as repeated addition and relate this concept to multiplication. Explain that $^{+}3 \times {}^{-}4$ means three negative fours $({}^{-}4 + {}^{-}4 + {}^{-}4)$, or three jumps of $^{-}4$ on the number line, as shown in Figure 17.18. The use of the model helps students make sense of the plus or minus sign in the answer. The problem $^{-}3 \times 4$ can be expressed, using the commutative property, as $4 \times {}^{-}3$ or four jumps of negative three. The problem $^{-}3 \times {}^{-}5$ can be thought of as taking 3 "backward" jumps of negative 5, resulting in an answer of $^{+}15$, as in Figure 17.19.

FIGURE 17.18
Multiplying integers using a number line.

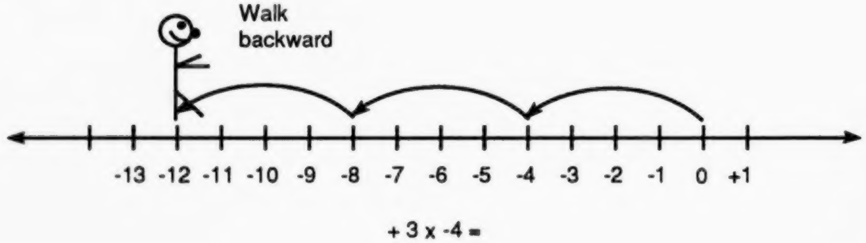
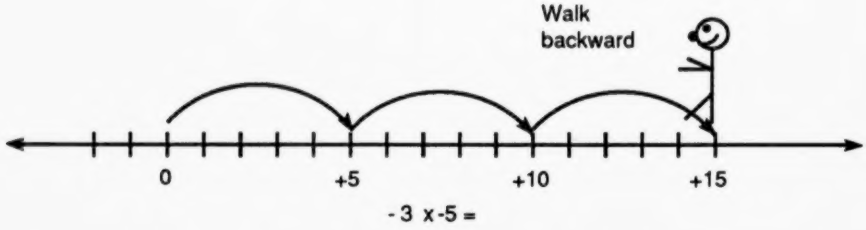


FIGURE 17.19
A number line for −3 × −5.



Numerical patterns can also be useful in developing a rationale for the sign placed next to the result of multiplying an integer: Positive × Negative = Negative/ Negative × Negative = Positive.

$5 \times 4 = 20$
$5 \times 3 = 15$
$5 \times 2 = 10$
$5 \times 1 = 5$
$5 \times 0 = 0$
$5 \times {}^{-}1 =$ ____
$5 \times {}^{-}2 =$ ____

$5 \times 4 = 20$
$5 \times 3 = 15$
$5 \times 2 = 10$
$5 \times 1 = 5$
$5 \times 0 = 0$
$5 \times {}^{-}1 =$ ____
$5 \times {}^{-}2 =$ ____

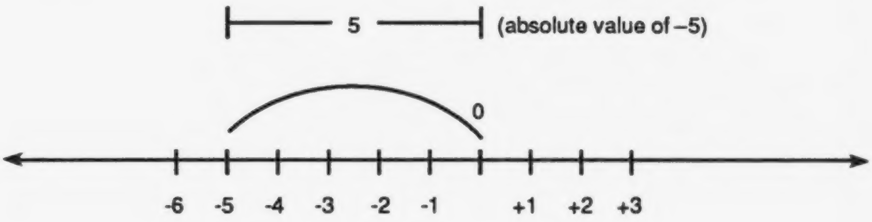
When teaching division of integers, remind students that division is the inverse operation of multiplication. If

$5 \times 4 = 20$
Then $20 / 5 = 4$
Likewise $5 \times {}^{-}4 = {}^{-}20$
${}^{-}20 / 5 = {}^{-}4$

Absolute Value

Present the concept of *absolute value* (the magnitude of a number). Practically, the absolute value of a number is its value, irrelevant of its sign. To help students understand this concept ask them to draw the arrow indicating the values ${}^{-}5$ and ${}^{+}7$ (as in Figure 17.20). Point out that the absolute values of ${}^{-}5$ and ${}^{+}7$ are the lengths of these arrows.

FIGURE 17.20
Showing absolute value on a number line.



■ Introduce students to the idea of a function in Logo or BASIC. Next, introduce the ABS function and ask them to write immediate (or delayed) PRINT statements to find the absolute values of several numbers. Next, have them solve problems like the following using pencil and paper, then write BASIC or Logo statements to check their work.

$4 + 4 \times | -7 |$
$-7 \times | +3 | / -7$

FIGURE 17.21
Prime numbers.

1	②	③	4	⑤	6
⑦	8	9	10	⑪	12
⑬	14	15	16	⑰	18
⑲	20	21	22	㉓	24
25	26	27	28	㉙	30
㉛	32	33	34	35	36
㊲	38	39	40	㊶	42
㊸	44	45	46	㊼	48
49	50	51	52	(53)	54
55	56	57	58	(59)	60
(61)	62	63	64	65	66
(67)	68	69	70	(71)	72
(73)	74	75	76	77	78
(79)	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
(97)	98	99	100		

Prime Numbers (Objective D)

The process of determining a number's factors is useful for finding common denominators and square roots. The *factors* of a number are those integers by which it can be evenly divided (i.e., the resulting quotient is a whole number). Factors of 12 are 1, 2, 3, 4, 6, and 12. *Prime numbers* are numbers with exactly two factors, 1 and the number itself. The number 1 is not prime because it has only one factor. Two is prime because it has two factors, 1 and 2. Eight is not prime because it has four factors, 1, 2, 4, and 8. Figure 17.21 shows the prime numbers between 1 and 100.

■ Give the students a table of numbers similar to Figure 17.21, but with no marks to indicate primes. Have them use their calculators to determine and mark the prime numbers. Ask children with advanced programming skills to write a program that finds all prime numbers between 1 and 1000.

Numbers with more than two factors, such as 8, 24, and 100, are called *composite numbers*. Any composite may be expressed as the product of prime numbers so the process of determining these primes is the process of prime factorization. One way to determine the prime factorization is to use factor trees. Figure 17.22 demonstrates this process.

■ Have students use their calculators and factor trees to find the prime factorization of a variety of numbers. Have them draw the trees as illustrated in Figure 17.22.

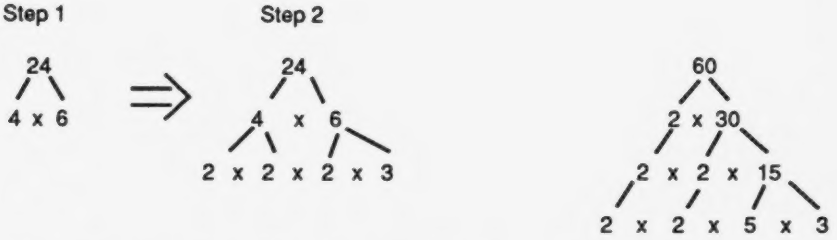


FIGURE 17.22
Factor trees.

Activities for Real Numbers (Objectives E–F)

Review the concept of *rational number.* Ask students to give a variety of examples of rational numbers. Next, draw a number line and mark two rational numbers (see Figure 17.23). Ask students to name a rational number that falls between those two numbers. Repeat for the lower original number and the students number. Ask for several more such numbers. Ask students how many numbers can be found between any two numbers on a number line.

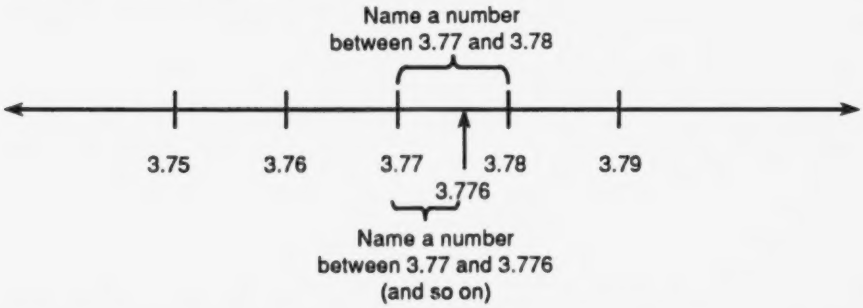


FIGURE 17.23
Real numbers on a number line.

Introduce one technique for finding a number between any two rational numbers: find their average. For example,

$$\frac{3.77 + 3.776}{2} = 3.773$$

$$\frac{3.77 + 3.773}{2} = 3.7715$$

and so on.

Discuss the idea that this process may be continued infinitely; thus, there are an infinite number of numbers between any two numbers, no matter how close they are to each other.

Introduce the concept of *real number*—any positive or negative repeating or nonrepeating decimal. Have students identify repeating and nonrepeating decimals (rational and irrational numbers, respectively). Present some of the repeating decimals as fractions.

EXAMPLE: Which of the following are irrational numbers?

1/7 .55555555. . . .33333

.050060007 .050050005. . . 3.14159. . .

and so on.

Introduce the concept of *square root*: the square root of a number is the number that, multiplied by itself, gives you the original number.

- Ask children to look at a multiplication facts table and identify numbers in the body of the grid (see Figure 17.24) that have two equal factors.* How many squared numbers are in each row? In each column? How many altogether? Why do you think the numbers 4, 9, 16, 25, 36, 49, 64, and 81 are called *squared numbers*? We say 7 is the *square root* of 49, because 7 × 7 = 49. What is the square root of 4? 25? 81? 64? 36? 100?

- Have students find the square roots of perfect squares like 49 and 25 by estimating the square root, then multiplying the estimate by itself to check their estimate. Have students estimate the square roots of nonsquares like 90. Have them use their calculators to check their estimates.

- Introduce the estimation and checking method for finding square roots of nonsquares on a calculator as discussed in Chapter 3. Have students find the square roots of several numbers to the nearest thousandth using this technique. After they have completed the activity, have them use a calculator with a square root function, the Logo or BASIC square root function, or a square root table to check their estimates.

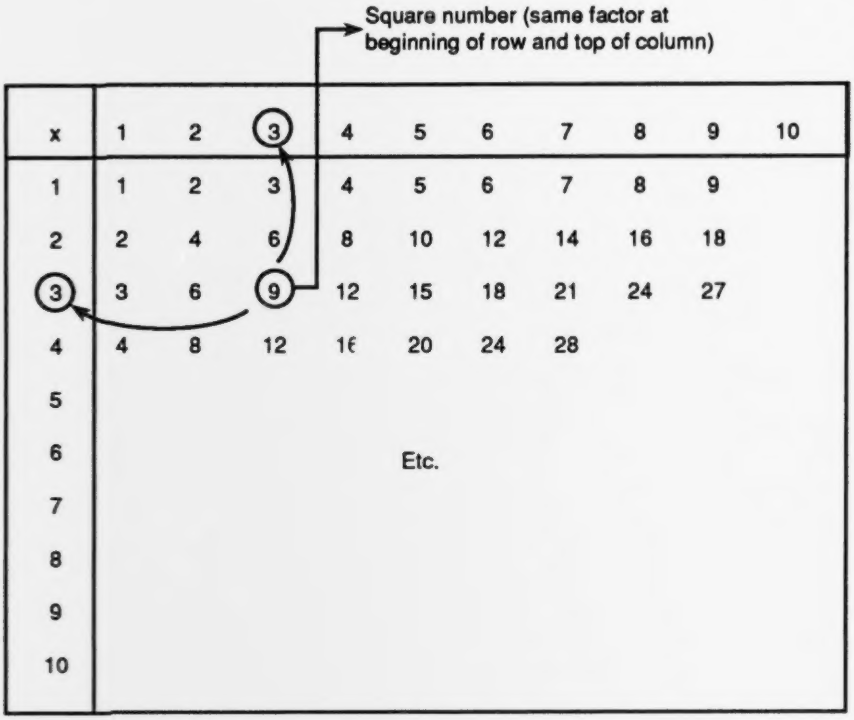
- Have the students use prime factorization to determine if numbers are squares, as in Figure 17.25. A number is square if its prime factorization contains two (or an even number) of each factor and it has no factors that appear only once (or an odd number of times).

- Review the Pythagorean theorem. Have the students find the hypotenuse of several right triangles using the Pythagorean theorem with the estimation and checking method for finding square roots. Have them check their answers by

---

*Activity contributed by Jewell Garner.

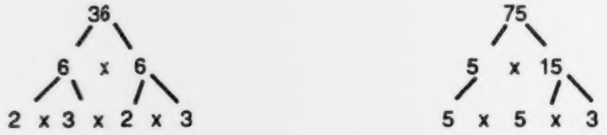
FIGURE 17.24
A grid for finding square roots.



Square number (same factor at beginning of row and top of column)

x	1	2	③	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	
2	2	4	6	8	10	12	14	16	18	
③	3	6	⑨	12	15	18	21	24	27	
4	4	8	12	16	20	24	28			
5										
6					Etc.					
7										
8										
9										
10										

FIGURE 17.25
Prime factorization.



Prime factorization has 2 twos and 2 threes—is a squared number

Prime factorization has only one 3—is not a squared number

drawing the right triangles in the problems and measuring, or give them a worksheet with the triangles already drawn.

- Have students find the distance between diagonal corners of the classroom or their desk using the Pythagorean Theorem (have them measure the lengths of the walls or sides of the desk), and the estimation and checking for square roots. Next, have them measure the diagonals to check their answer.

SUGGESTED READINGS

Ashlock, R. B., and West, T. A. Physical representations for signed-number operations. *Arithmetic Teacher* 14(7), November 1967: 549–554.

Cohen, L. S. A rational in working with signed numbers. *Arithmetic Teacher* 12(7), November 1965: 563–567.

Dirks, M. K. The integer abacus. *Arithmetic Teacher* 31(7), March 1984: 50–54.

QUESTIONS

1. Describe an analogy using sea level for developing the concept of negative integer.
2. Write a Logo program that models what happens when you add or subtract integers (first draw a number line, then position the turtle and move it to model the operation).
3. Use factor trees to find the prime factorization of 1050 and 1260. Use prime factorization to find a square number larger than 500.
4. Use the estimation and checking method with a calculator to find the square root of 69 to the nearest thousandth.
5. Write a BASIC or Logo program that uses trial and checking to find indirectly (i.e., *not* by using the built-in square root function) the square root of any number. Hint: Set up a loop that divides the input by a constant, squares the result, and if the resulting number is larger than the original does one thing and if it is smaller does something else. Compare the square roots obtained using your method to those of someone else in the class. Whose algorithm is more accurate? Why?

CHAPTER

18 Data Gathering, Statistics, Probability

In most of the mathematical situations discussed so far in this book, the number of values operated on by students is limited to two or, at most, three or four numbers. In real life, however, we must often make sense out of dozens, hundreds, thousands, or even more numbers. Witness the number of articles that appear in newspapers and magazines describing how people (e.g., those in different age groups) feel about certain political issues, brands of merchandise sold, or stock market trends. Without the ability to understand statistical and probability notions and read graphs, a person would have little understanding of the general trends in our society: he or she would not be literate.

TEACHING STATISTICS, GRAPHING, AND PROBABILITY

Statistics (including graphing) and probability are important tools that help us explore, understand, and manipulate our environment. Statistics can help a child understand how plants grow and propagate or how students in the fifth grade feel about undertaking a certain class project, or determine which choice to make in a game of Yatzee or cards.

Behavioral Objectives for Statistics, Graphing, and Probability

A. Make a bar graph using direct comparison.
B. Make a bar graph using counting markers from real data.
C. Make a frequency table from real data.
D. Make a grouped frequency table from real data.
E. Make a histogram from real data.
F. Interpret and discuss trends from a histogram.
G. Find the range from data.

H. Find the mode from data.
I. Find the median from data.
J. Find the mean from data.
K. Use a spreadsheet to organize and evaluate data.
L. Gather data to investigate a question and use appropriate graphs and/or measures of central tendency to investigate that question.
M. Given the total number of events and the number of favorable events, state the probability that an event will occur.

Teaching Graphing (Objectives A–F)

When categorizing data, looking at trends, or comparing groups, we use statistical procedures rather than simple, single operations. Statistics are mathematical procedures used to describe and make inferences about groups of individuals, animals, events, or objects. Often times a single number or a graph will be used to describe a group of hundreds of individuals or several groups of individuals. If we wanted to describe the height of women in a tribe living in Papua, New Guinea, for example, we might find the mean height of the women or summarize the data in a graph rather than listing the height of every woman in the tribe. If there are dozens, hundreds, or thousands of numbers describing some aspect of a group, we have trouble comprehending the meaning of the data. For example, if we wanted to determine whether students in Wilson Elementary are more overweight than students in Hasting School, simply listing the weights of all children in the two schools would not provide an answer.

Alternatives to listing individual values for each person or object in a group are (1) to group the data in some way and to put the data into graphs or tables, or (2) to give a single value (measures of central tendency) that describes the group.

In addition to descriptive statistics which use data from each individual in the group we are studying (as in finding the average weight of children in Mrs. Jones's third grade class), we can use inferential statistics, wherein we select and study a small portion of the group we are examining in order to make inferences about the larger group. Many times it would be difficult or impossible to gather all the data (numerical information) needed from a large group. For example, it would be nearly impossible (and extremely expensive) to poll everyone in the United States to determine how many people watched *Miami Vice* on a particular evening. Rather, we obtain a sample of a few hundred viewers and make inferences about the viewing population (the group being studied).

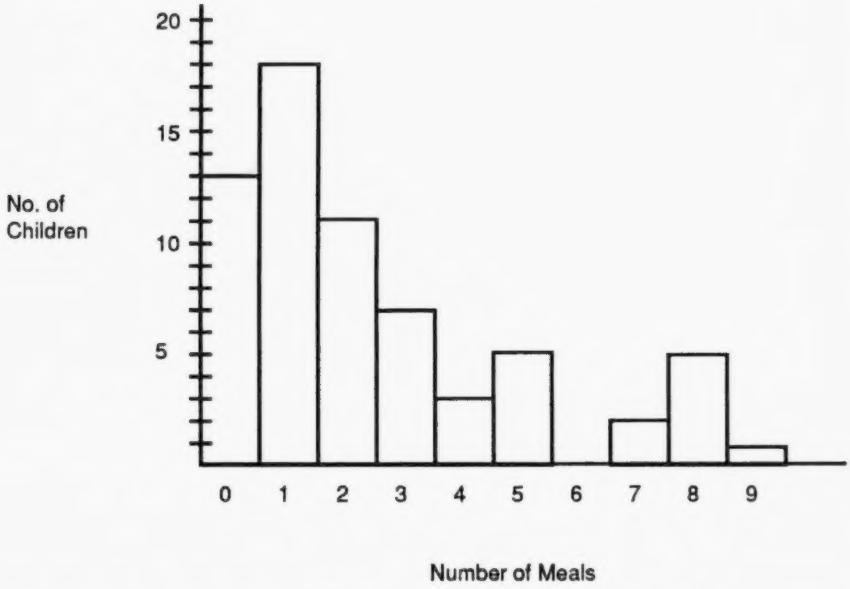
Suppose that you were teaching in a suburban, middle-class school and, in conjunction with a unit on nutrition, decided to find out how many restaurant meals children in the fifth grade eat during a given week. You might start by asking the children to write how many times each ate out (this could even be expanded to include the types of restaurants and the items included in the meals). The next step would be to put the data in a table with a column for tally marks for the number of children in each category, and another column containing the totals, as in Figure 18.1.

Once the data have been organized in a table, they can be represented in a graph. Graphs present pictures of data so that trends and groupings in the data can be seen at a glance. A *histogram* or *bar graph* consists of bars of equal width whose height or length corresponds to frequencies being represented. Figure 18.2 is a bar graph based on the data in Figure 18.1. The horizontal axis represents the number

FIGURE 18.1
A frequency table.

Number of Rest. Meals		Number of Children
0	𝍸𝍸|||	13
1	𝍸𝍸𝍸|||	18
2	𝍸𝍸|	11
3	𝍸||	7
4	|||	3
5	𝍸	5
6		0
7	||	2
8	𝍸	5
9	|	1

FIGURE 18.2
A bar graph.



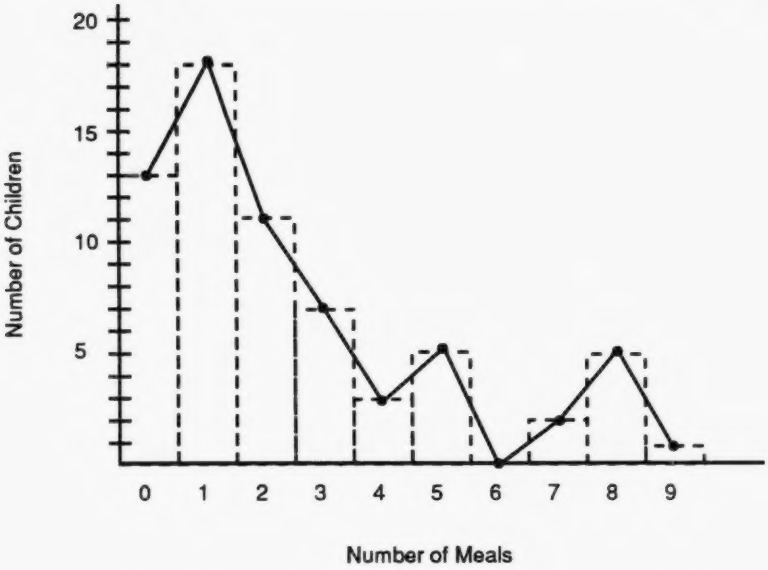
of restaurant meals eaten, from 0 through 9. The vertical axis represents the number of children in the categories and the height of the bars corresponds with the number on the axis.

Note that in a graph, each axis is labeled and the graph has a title. Also note that, whatever an axis represents, the *unit of distance* along its length is consistent (i.e., if one centimeter on the axis is chosen to represent 5 occurrences, this scale is used along the entire axis). On both the horizontal and vertical axes, each possible number from the smallest in the data to the largest in the data is represented, even if there were no occurrences in that category (e.g., no children ate that many meals out). Bar graphs are sometimes drawn so that the bars are horizontal rather than vertical and the length, rather than the height, of the bars represents a given quantity. (See Figure 18.4 below for an example.)

A ***frequency polygon*** or ***line graph*** is similar to a histogram or bar graph, but consists of a broken line connecting the midpoints of the tops of the bars in the corresponding bar graph (histogram). Figure 18.3 shows a frequency polygon for the data in Figure 18.1.

FIGURE 18.3
A line graph or frequency polygon.

Number of Restaurant Meals Eaten by Fifth Graders at Wilson School 6/5 to 6/12



Even though bar and line graphs are sometimes used interchangeably, bar graphs are usually used when the data are categorical (fit into clearly-defined categories, e.g., days of the week, hair colors, etc.), while line graphs are used with continuous data (height in centimeters, age, etc.). Figures 18.4 and 18.5 show further examples of bar graphs and line graphs.

One of the best things about teaching graphing and statistics is that so many interesting activities involving real data gathering can be planned. It is easy to find sources for data involving the children themselves, their families, television programs, sports, or hobbies. Children enjoy gathering this type of data. Try to involve them as much as possible in planning and gathering, as well as in analyzing the data, just as you would when teaching a science lesson. Of course, you should avoid gathering information that might cause embarassment to them or their parents.

The first exposure to graphing should involve bar graphs where the height of the bars is determined by the actual length of objects or by stickers or marks corresponding one-to-one with real objects.

FIGURE 18.4
Examples of bar graphs.

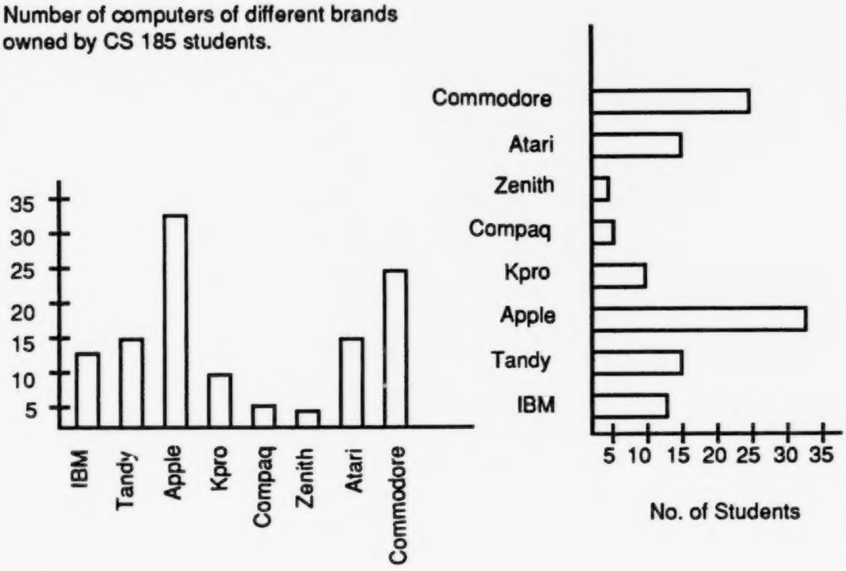
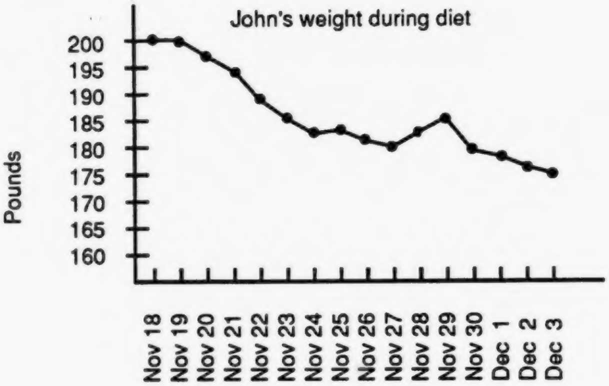


FIGURE 18.5
Another example of a line graph.



- An activity for kindergarten or first grade children involves a graph of handspans. Use a yard-long sheet of butcher paper. Draw a horizontal axis, then a vertical axis with centimeter markings. You may also wish to extend the centimeter markings lightly with pencil parallel to the base line so that the children may read the length of their span from the graph. Have each child place the tip of his thumb on the base line, stretch his hand toward the top of the graph and draw around his thumb and little finger using a crayon or felt marker. Next, have him connect the lines to make a "bar" (see Figure 18.6). Discuss hand placement so that hand lengths can be accurately compared (i.e., the "bars" should all be parallel). Finally, have him write his name below his span. After all children have graphed their handspans, discuss the resulting "graph" with them. Whose span is the largest? Whose is the smallest?

- Another type of graph used to introduce graphing is one with counters or pictures in one-to-one correspondence to the value being measured. Many teachers

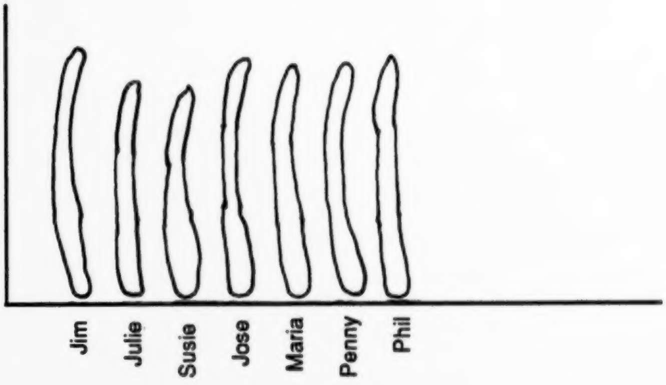


FIGURE 18.6
A direct comparison bar graph.

use grid paper and stickers to record and display when children have completed assignments. With a little forethought and a proper introduction and discussion, this display can be turned into a graphing activity, as in Figure 18.7. Instead of simply placing a star in a row or column on the grid after a child's name, use square stickers that fit together to form a bar or place the star on the chart, then color in the corresponding square to form a bar. (Of course, there should be no gaps in the bars.) The height of the bars corresponds to the number of times an activity has been completed but a particular square on the chart will not correspond to a particular assignment or date. Leave a little room between the bars so that each is distinct from the others.

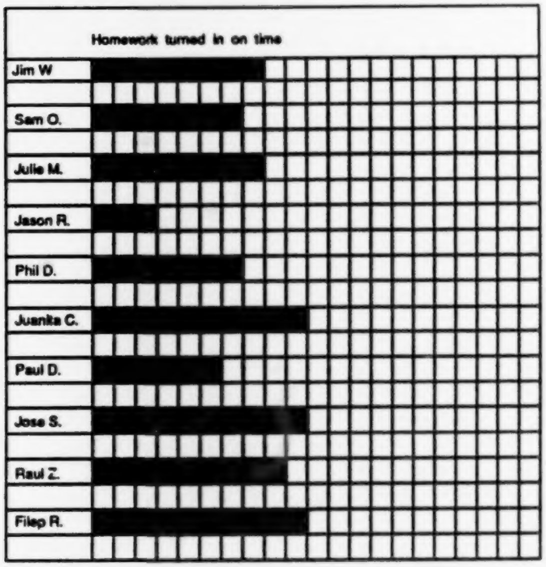
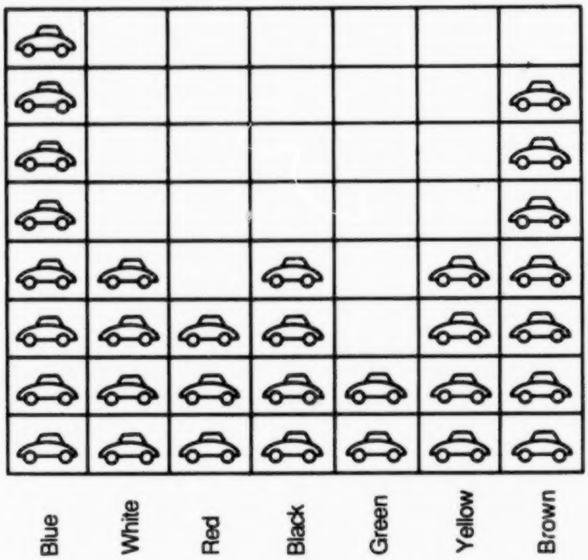


FIGURE 18.7
A bar graph used for recording assignment completion.

- Make a count of the different colors of automobiles owned by families of children in the class. Give children a dittoed sheet with small pictures of cars and have them use crayons to color the cars—one for each car of each color in the class. Have them cut them out and paste them in columns by each color on grid paper, as in Figure 18.8.

FIGURE 18.8
A picture graph for car colors.



■ Have groups of 3 or 4 students set up data-gathering activities, then make frequency tables to organize their data. Using the tables, have them draw frequency histograms or bar graphs. You might wish to give them a list of appropriate activities and allow them to choose the one that they find the most interesting. Some possibilities follow:

1. Write the names of five television shows on a piece of paper and have students mark the one they prefer.
2. Find out how many children in the two third-grade classes have brown, blond, black, or red hair.
3. Find out how many uncles, aunts, first cousins, and second cousins students in the two third-grade classes have and graph the results.

Introduce and discuss line graphs for continuums (as distinct from the frequency polygon that charts categorical values). The main difference between bar graphs and these line graphs is the fact that with line graphs, the horizontal axis reflects an underlying continuum of values rather than categories. Even though discrete (i.e., noncontinuous) points are plotted on the graph, we assume that the lines connecting the points illustrate regular increases or decreases between the points.

■ Have students gather data and make graphs using some continuous variable such as the growth of bean sprouts. For example, your students could plant beans in styrofoam cups and water them a given amount each day and record the plants' heights at the same time for two weeks. Have them record the data in a table and graph the data using line graphs, with the verticals axis representing the height of the plants. You might want to vary the light and/or the amount of water given to different plants, then compare and discuss the resulting graphs.

■ Have students record the age in months of everyone in the class. Next, have them categorize the data. When setting up the intervals, start the left-most interval with the youngest person in class and make the other intervals all the same width (e.g., 120–122 months, 123–125 months, and 126–128 months, and so forth).

There should be between 8 and 15 intervals. Have students record the results in a grouped frequency table, then graph the results using a frequency polygon with the month groupings on the horizontal axis and the frequencies on the vertical axis.

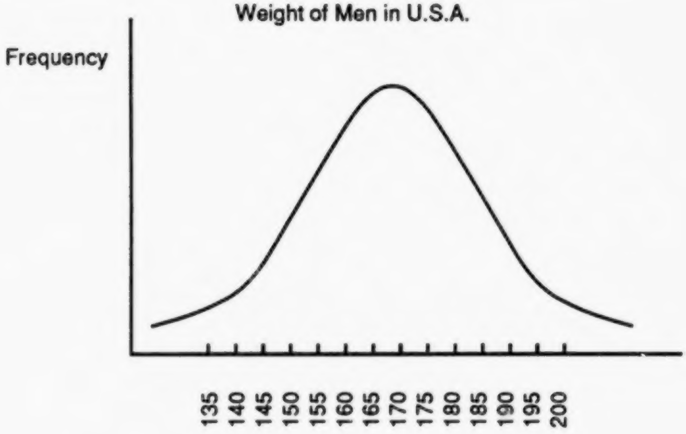
Activities for Descriptive Statistics (Objectives G–K)

Descriptive statistics, for the most part, describe a group with a single number. Any single value used to describe dozens, hundreds, thousands, or more individuals will be an incomplete description, just as a single adjective like *American, intelligent, female,* or *Chicano* is an incomplete description of a person or group of people.

When using statistics to describe a group, as when using adjectives or adverbs to describe a group of people, the type of statistic used depends on the situation. The *mean, median,* and *mode* each describe the middle of a group, in a different way and range. The *range* is one way to describe how spread out the scores are.

The **mean** score is determined by adding together all the values and dividing by the number of values. The mean is used to find the middle of a group of values when the underlying distribution of numbers is "normal," that is, symmetrical and bell shaped. A normal distribution, pictured in Figure 18.9, has most of the scores bunched around the mean and approximately the same number of scores above as below the mean.

FIGURE 18.9
Normal distribution.



A formula for finding the mean is as follows: $\frac{\Sigma x}{N}$

Sigma (Σ) means "the sum of"; **X** means the individual scores; **N** is the number of scores. To find the mean of the quiz scores 2, 6, 5, 7, 4, 5, 6, 9, add them up (**ΣX**) and divide by the number of scores, (**N**), 8.

$$\frac{2 + 6 + 5 + 7 + 4 + 5 + 6 + 9}{8} = 5.5$$

The **median** (middle score) is often used when a distribution is skewed, unbalanced on one side of the middle. For example, government statisticians usually refer to the median income of people in the United States rather than the mean because the distribution is skewed by the presence of rich people making millions and even billions of dollars. The mean or average income (determined by adding the income of everyone in the U.S.A. and dividing by the population) would be higher than what the average or middle person makes—the median amount. The median presents a more accurate picture in such a case. ***The median is determined by arranging all scores in ascending or descending order and identifying the score that is half way between the extremes.*** If there is an odd number of score, the middle score is the N/2 + .5th score. For example, if there are 25 scores, the middle is the 25/2 + .5 or 13th score from the bottom or top of the distribution. If there is an even number of scores, the middle score is halfway between the N/2 and the N/2 + 1st score. For example, if there are 12 scores, the median is the average of the 6th (12/2) and the 7th (12/2 + 1) scores.

The **mode** is the most frequent score in a distribution. In a normal distribution, the mode corresponds with both the mean and the median. In skewed distributions, however, it is usually at a different place and may even be the smallest or largest score. Suppose, for example, you give a very easy test and almost half the class gets all the answers right. Your frequency table and bar graph might look like the following:

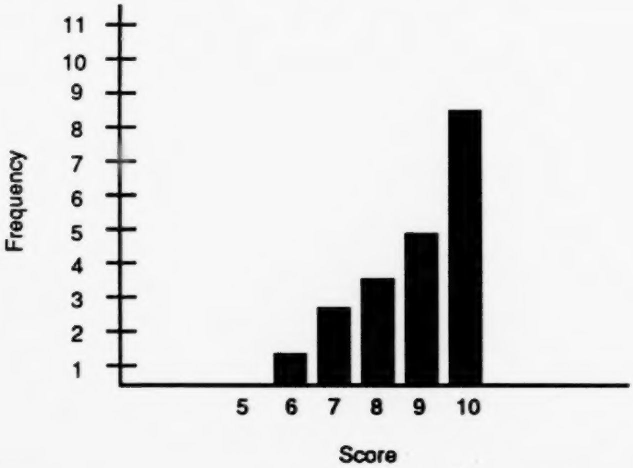
Score	*Frequency*
10	11
9	6
8	4
7	2
6	1
5	0

The mode in this distribution is 10. The mean of the distribution is found by adding all the values

$$\frac{(11 \times 10) + (6 \times 9) + (4 \times 8) + (2 \times 7) + (1 \times 6)}{24} = \frac{216}{24} = 9$$

Note that the mean (9) is near the highest score in this distribution.

FIGURE 18.10
Skewed distribution.



Since there are 24 scores (11 + 6 + 4 + 2 + 1) in the distribution, the median is the average of the 12th and 13th score. If all the scores are listed from smallest to largest, as below, the middle two scores are both 9; therefore the median is 9 (the average of 9 and 9 is $\frac{9 + 9}{2} = 9$).

Score #:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Score :	6	7	7	8	8	8	8	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10	10	10

middle scores

Another way to describe a set of data is to consider the *range* of a distribution: that is, the distance between the bottom and top scores. The range of scores in Figure 18.10 is 10 − 6 (since no one scored 5 or below) or 4. The range can be useful in describing a group or in distinguishing between two groups. For example, suppose you teach two classes of sixth grade mathematics and give the same chapter test to each. Class 1 has a mean score of 78 and Class 2 a mean of 79.1 and from this standpoint they seem much alike. However, if the range of scores for Class 1 is 10 and for Class 2 is 45, there is an obvious difference between the two groups.

- Have groups of students gather data about the eating habits of students in their class. One group may want to research the number of snacks students have a day (perhaps on Saturdays), the number of bottles or cans of soda they have a week, the number of times they eat breakfast a month, the number of desserts they have a week, and so forth. Have them find the mean, median, mode, and range for each. Have them graph the results, and compare different groups. This activity would be good to integrate with health lessons.

- Have students find the mean height of students in the class. First, set up a station where students can measure their height in centimeters. Make a form for them to record the heights. As they calculate the mean from the data, teach them how to use the memory keys on their calculators to store intermediate results. Finally, have them compare results.

- Have students write a BASIC or Logo program that will allow the user to enter any number of values, then find the mean of those values. Ask them to use the programs to find the mean from real experiments such as those discussed above.

- Repeat for the median, mode, and range.

Activities for Probability (Objective L)

A number stating the likelihood that a particular event will occur is called a *probability*. Probabilities are expressed as fractions (or the equivalent decimal or percentage) with numerators indicating the possible ways an event can occur and denominators indicating the total number of possible events. For example, if you roll a die, the number of ways you can roll an even number is 3 (you can roll a 2, 4, or 6) and the total number of possible numbers that can turn up is 6. The probability of rolling an even number, then, is 3/6 or .5.

There are two types of probability that we can present to children—empirical probability and theoretical probability.

Empirical probability is a value that results from experimentation.

For example, if you had a brown paper bag filled with coins and drew out 20 of them—13 pennies, 5 dimes, and 2 nickels— you might conclude that the probability of drawing a penny out of the bag on the next draw would be 13/20; that is, if you drew 20 more coins without looking, 13 would be pennies. Or, if a baseball player has had 132 hits out of 400 at bats, you might conclude that the probability of him getting a hit in the future would be 132/400 or .330 (his batting average).

> *Probabilities obtained without experimentation, based on the prospects of certain events occurrring or not occurring, are called theoretical probabilities.*

You might conclude that the chances of throwing a head when flipping a coin would be 1/2 because a coin is symmetrical and there is only one way to obtain a head but two possible outcomes—a head or a tail. In practice, if you flip a coin 100 times, you might obtain 46 heads and 54 tails, but this wouldn't change the theoretical probability.

Beginning activities with children should involve empirical probabilities. Have them gather and record data on tally sheets. You may also wish to graph the results and find medians, means, and so forth.

- Give pairs of children two dice each and a tally sheet. Have one child roll the dice and have the other record the sum of the dice on a tally sheet. Have the children draw a bar graph depicting the results. Have the students compare their results with those of other children in the class.

- Have children do a sampling activity where they determine the probability of an event occurring, then have them take another sample and see how accurate their prediction was. For example, a pair of students could stand in the school hallway for 5 minutes at break time and record the number of people who are wearing red and the total number of people who pass. What is the probability based on this sample that a person passing by will be wearing red? Repeat at another time or in another part of the school. Are the results the same?

- Make a wooden frame one foot square. Place the frame on the ground in the school yard or parking lot 50 times at random and record the number of times that there are one or more ants in the frame. (You will need to impress upon students that the placement of the frame should be random and that they should not look for ants, then place the frame at the spots where they are.) What is the probability that there will be an ant in a foot-square location anywhere on the playground?

- Have students determine the theoretical probability of rolling a total of 5 with a pair of dice. Let them roll the dice 100 times and record the number of fives that are rolled. How close to the theoretical probability is the empirical probability?

- Ask your students to watch the weather report on television and record the probability of rain that is given each night. Discuss what this means and observe what happens on several occasions when the predicted probability of rain is high. You may wish to chart the predictions and whether or not it does rain. After a month or two, see if on the occasions when the weather forecaster predicted, for example, a 20% chance of rain it rained on about 20% of the following days.

SUGGESTED READINGS

Burns, Marilyn, Put some probability in your classroom. *Arithmetic Teacher* 30 (7), March 1983: 21–22.

Christopher, L. Graphs can jazz up the mathematics curriculum. *Arithmetic Teacher* 30 (1), September 1982: 28–30.

Horak, V.M. Collecting and displaying the data around us. *Arithmetic Teacher* 30 (1), September 1982: 16–20.

Huff, D. *How to Lie with Statistics*. New York: Norton, 1954.

Johnson, E.M. Bar graphs for first graders. *Arithmetic Teacher* 29 (4), December 1981: 30-31.

Mathematics for the Middle Grades (5–9) (1982 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: The National Council of Teachers of Mathematics, 1982.

O'Neal, D.R., and Jensen, R. Looking at facts. *Arithmetic Teacher* 29 (8), April 1982: 12–15.

Shulte, A.P. and Choate, C.A. *What Are My Chances?* Palo Alto, CA: The National Council of Teachers of Mathematics, 1979.

Teaching Statistics and Probability (1981 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1981.

Woorward, E. A second-grade probability and graphing lesson. *Arithmetic Teacher* 30 (7), March 1983: 23–24.

QUESTIONS

1. Gather data on the number of math classes (excluding math methods classes) that students in your class have had since junior high school. Graph the results. Find the mean, median, mode, and range.

2. What does the standard deviation tell us about the scores in a group? Should standard deviation be taught in the elementary grades? Why or why not?

3. Describe how you might make a transition from bar graphs to line graphs with your students.

4. Write a Logo program that draws a graph for the data in Figure 18.7. Advanced assignment: write a Logo or BASIC program that allows the user to enter given values, then draw a bar graph using these values.

5. Write a BASIC or Logo program that will allow the user to enter any number of values, and that will then calculate the mean, median, mode, and range of these values.

CHAPTER

19

Assessment, Planning, & Management

The biggest hurdles facing beginning teachers usually do not relate to the content to be taught. Instead, they involve the day-to-day operation of the classroom—determining what students know and do not know, planning lessons, delivering them, organizing students so that individual needs are met, and, on top of everything, moving materials and media in and out of the lessons with ease. A beginning teacher often feels like someone who buys an unassembled bicycle without any instructions for assembly. All the pieces are there, but where do they all go?

When considering the organization of this text, the author was torn between placing this chapter at the beginning to emphasize its importance, or near the end of the text, after the student has examined what to assess and manage. As you see, the latter prevailed; however, note that this is one of the most important and fundamental chapters in the book.

ASSESSMENT

Suppose that, unfit and slightly overweight, you sign up for an exercise class. On the first day of the class the instructor puts you in a group with others in your age group and states, "The statistics show that the average 25 to 30 year old can do 10 minutes of continuous aerobic exercise, so that is where we are going to start. In three weeks, we will be up to 20 minutes and in six weeks 30 minutes." Actually, most people in the group would head for the door upon hearing such an introduction. Children who encounter similar experiences cannot opt out of their schooling or move to a class where instruction is at their level; yet, unfortunately, this is the way they are often treated. We use statistics of average achievement such as the results of standardized tests to determine what they are taught.

A better strategy for an aerobics instructor is to assess the fitness level of each participant, then plan a fitness program based on his or her need. To do otherwise is to risk causing injury or even death on one hand, or having an exercise program that is not demanding enough to improve participants' fitness on the other. In education, since no one is average, we need to determine students' levels of knowledge, capabilities, and/or learning handicaps and plan instruction accordingly. To do otherwise is to risk emotional distress or failure on the one hand or boredom and lack of progress on the other.

One aspect of assessment relates to the subject being taught—in mathematics we might assess a student's readiness to learn decimal numeration, her mastery of the material in a just-completed unit on metric measurement of length, her understanding of general mathematical principles, or her thought processes when she estimates products of multidigit factors. Other aspects of assessment could include evaluating a child's psychological state (e.g., her attitude toward mathematics), her I.Q., or her manual dexterity.

Assessing Readiness

The importance of determining a child's readiness to learn a mathematical topic cannot be overestimated. It is obvious that if a child cannot subtract multidigit numbers, he will not be able to divide using the long-division algorithm. Other readiness skills and concepts related to long-division are the ability to estimate and calculate products and the understanding of place value. This type of readiness is called *content readiness*. Content readiness involves the skills and concepts that are prerequisite to a particular mathematical task or idea. Following is a listing of the areas of content readiness for multidigit subtraction.

- Place value and numeration
- Regrouping
- Meanings of subtraction
- Recognition of subtraction symbols

Assessment of children's content readiness may involve any or all of the strategies discussed below. Probably the most beneficial are individualized diagnostic tests and teacher-made diagnostic tests since they give the needed detailed information about the child's performance in specific content areas. Of course, after the child's readiness has been determined, it is up to the teacher to plan instruction to meet the needs discovered.

Types of Assessment

Sometimes teachers think of assessment as involving only pencil and paper tests. However valuable pencil and paper tests are, they represent only one relatively limited type of assessment. Certain children may be able to demonstrate a mathematical operation like subtraction with objects or use it when money is involved, but have difficulty finding answers to pencil-and-paper subtraction problems. On the other hand, some children will be able to obtain correct answers to their pencil and paper computations but have no understanding of what they are doing or how to apply these techniques in real-life settings. In order to obtain a more complete evaluation of a child's strengths and weaknesses in mathematics we should include not only tests but also evaluation of assignments, observation, and interviews.

1. Tests: Tests come in a large variety of shapes and sizes and are designed for a variety of purposes. **Standardized achievement tests** are designed to give teachers and administrators a general idea of students' achievement in a particular area.

These tests are normed, meaning that they have been administered to many children to determine what scores are normal for children at each appropriate grade level. If a child scores at the 89th percentile in math applications on a particular standardized achievement test, that means that she scored higher than 89 out of 100 children in the norming population at her age level on that portion of the test. That might translate to a grade equivalent of 3.0 for a child who is in the second grade. These scores give a rough idea as to where the child stands in relationship to other children, but give little information about the actual strengths and weakness of the individual child. Even though the chid is working "above grade level," she may have some severe deficiencies in important areas of mathematics that are counterbalanced by strong performances in other areas.

Individualized diagnostic tests give much more information about a child's strengths and weaknesses. They are administered by the teacher or diagnostician who selects items that are more or less at the student's level (avoiding the problem inherent with group standardized tests where children face many problems that are either too easy or too difficult). Scores on these instruments are usually more accurate than those obtained with group tests since they are verbally administered and poor readers are not penalized. Also, a student with problems is not as intimidated by this test since he only sees one problem at a time and since most of the problems are at his level. One of the most widely used individualized diagnostic test is *Key Math Diagnostic Arithmetic Test* (Connolly, et al., 1971). *Key Math* provides achievement levels in numeration, addition, subtraction, multiplication, division, fractions, problem solving, measurement, and money.

Even an individualized diagnostic test, however, will not provide detailed information about a child's strengths and weaknesses. For example, *Key Math* will not tell you which mutiplication facts the pupil does not yet know or why he is having difficulty adding fractions. This type of information is provided by **teacher-made diagnostic tests.** Typically, a teacher will design such a test after having decided which area she needs more information about (e.g., multiplication of fractions). This may be based on the child's performance on daily work or after having given and scored an individualized test like *Key Math*.

When developing the test, the teacher selects or designs several test items covering each precise skill. For example, if the child is having problems involving multidigit subtraction with regrouping and the teacher has determined that he understands the meaning of subtraction, knows his subtraction facts, and understands numeration concepts, she might ask the student to use manipulatives to model several problems involving subtraction of three-digit numbers with regrouping to the ones' place, then to the tens' place. Next, she might have him do several more problems with regrouping in both places. If the student is successful with the materials, she would have him do three pencil and paper problems for each skill (regrouping to the ones' place, regrouping to the tens' place, and regrouping to both places). Figure 19.1 shows an example of such a test. Note that items testing each skill (regrouping to ones' place, regrouping to tens' place, and regrouping to both) are randomly arranged in the test.

Teacher-made diagnostic tests should not be confined to paper-and-pencil tasks. They may cover any area of the curriculum, including problem-solving skills, estimation and mental arithmetic, and calculator skills.

2. Interviews: Another way to determining a child's problem areas in math is to listen to what she has to say. You may ask the child why she is unable to complete a task or where she is having difficulties. Another possibility is to ask the child to

FIGURE 19.1
A teacher-made diagnostic test.

Name ____________________
Date ____________________

1. $\begin{array}{r} 506 \\ -\,231 \\ \hline \end{array}$
2. $\begin{array}{r} 412 \\ -\,208 \\ \hline \end{array}$
3. $\begin{array}{r} 342 \\ -\,126 \\ \hline \end{array}$
4. $\begin{array}{r} 736 \\ -\,387 \\ \hline \end{array}$
5. $\begin{array}{r} 300 \\ -\,216 \\ \hline \end{array}$
6. $\begin{array}{r} 429 \\ -\,337 \\ \hline \end{array}$
7. $\begin{array}{r} 596 \\ -\,339 \\ \hline \end{array}$
8. $\begin{array}{r} 614 \\ -\,473 \\ \hline \end{array}$
9. $\begin{array}{r} 522 \\ -\,163 \\ \hline \end{array}$

"think out loud" while she is working on a specific task or ask her to explain what she did after each step.

3. Observation: We can learn much by watching a child during specific tasks in mathematics. Is the child engaging in avoidance behavior? Does he appear to be confused, making many corrections and changes his work? Does he apply himself immediately and confidently to a task? Does he attempt to obtain answers from others or continuously verify his own answers by checking other people's work? Observation of children during testing, seatwork, or group work situations can provide much information about their self-confidence, anxiety, or enthusiasm toward mathematics. You may wish to regularly observe each child in your class for a fixed period of time (e.g., 10 minutes) and take notes on his or her work habits or areas of difficulty.

4. Evaluation of assignments. Sometimes we are tempted to simply grade students' assignments and return them with a grade. However, children's paper work may provide us much information about erroneous thought processes if we use "error analysis."

Error Analysis

Error analysis is the process of looking for patterns in childrens' written work and determining their cause. A look at the digits "carried" in the problems below provides a clue about the child's thinking process. Apparently he thinks that 7 x 8 is 54:

1. $\begin{array}{r} {}^{5}17 \\ \times\ 8 \\ \hline 134 \end{array}$
2. $\begin{array}{r} {}^{5}28 \\ \times\ 7 \\ \hline 194 \end{array}$
3. $\begin{array}{r} 80 \\ \times\ 7 \\ \hline 540 \end{array}$

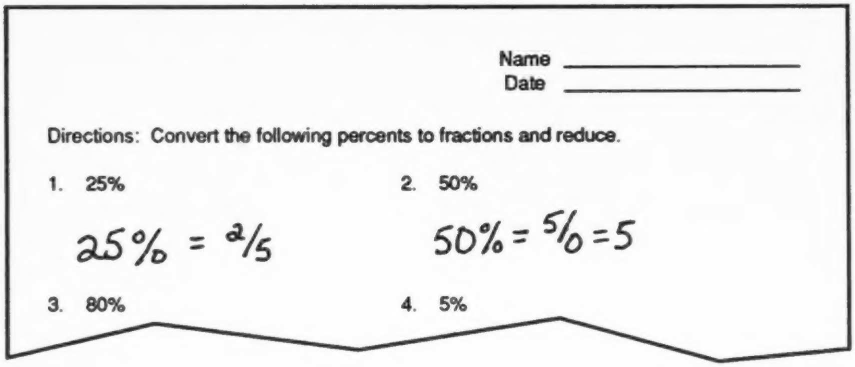
Research by Engelhardt and Wiebe (1979) identified 5 categories of errors made by children:

Procedural errors: These are errors derived from systematic and inappropriate algorithmic procedures. For example, a child might consistently multiply by the reciprocal of the dividend rather than the divisor in division problems with fractions:

1. $3/4 \div 1/2 = ?$ $4/3 \times 1/2 = 4/6 = 2/3$
2. $5/6 \div 2 = ?$ $6/5 \times 2 = 12/5 = 2\,2/5$
3. $1/2 \div 2/3 = ?$ $2/1 \times 2/3 = 4/3 = 1\,1/3$

Conceptual errors: These errors arise from incomplete, absent, or incorrect understanding of underlying mathematical concepts and principles. For example, a student may incorrectly convert percentage numbers to fractions because he does not understand the meaning of percents. (See Figure 19.2.)

FIGURE 19.2
A worksheet to find conceptual errors in a student's work.



Name ____________________
Date ____________________

Directions: Convert the following percents to fractions and reduce.

1. 25%

$25\% = 2/5$

2. 50%

$50\% = 5/0 = 5$

3. 80%

4. 5%

Mechanical/perceptual errors: These systematic and nonsystematic errors result from visual or motor-related difficulties such as illegible number formation, misalignment of numbers, or miscopying or misreading digits or symbols. Of course, many procedural errors (like those illustrated below) might also be considered to be, to some degree, conceptual errors. For example, a student might give an incorrect solution to a multiplication problem because she sloppily wrote and then misread a digit:

1. $$\begin{array}{r} 15 \\ \times\ 13 \\ \hline 35 \\ 12 \\ \hline 153 \end{array}$$

2. $$\begin{array}{r} 25 \\ \times\ 12 \\ \hline 40 \\ 25 \\ \hline 270 \end{array}$$

Basic fact errors: These errors result from an inaccurate automatic recall of basic number facts.

Careless/random errors: These nonsystematic errors result from momentary inattention to relevant stimuli. For example, a pupil might add when subtraction is called for simply because he was not observant of the operation sign.

Any written work by your pupils can be a source for observing error patterns. The best sources, however, are teacher-made diagnostic tests and pencil-and-paper practice pages since students will have multiple examples of specific tasks. If you are interested in pursuing the topic of error patterns further, you might wish to read *Error Patterns in Computation* (Ashlock 1983). This excellent book helps teachers discover and deal with children's error patterns in arithmetic.

Nonmathematical Causes of Failure

In any typical classroom setting, certain children will have difficulties with "grade-level" math. Some of these children's problems relate to lack of content readiness as discussed above. For others, the reasons may be psychological or perceptual. Teachers may be able to discover and deal with these in their classrooms or they may need to refer the child to the diagnostic specialist or school psychologist for diagnosis and treatment. A brief discussion of non-mathematical causes for failure follows:

Physiological causes: If the child is physically abused, undernourished, ill, or under the influence of drugs, learning and performance will be impaired.

Psychological causes: In order for learning to occur, the child needs to feel safe, secure, and positive about himself. Disruptive behavior, withdrawal, or failure to try can be signs of a psychological need.

Perceptual difficulties: Some children have difficulty understanding pictorial representations of mathematical situations. For example, they may not interpret a picture of 5 ducks on a pond with 3 more walking down the hill to join them as the addition problem "5 + 3." Others may have difficulties reading numbers or mathematical symbols. Of course, dyslexia (impairment of the ability to read) can affect children's mathematical achievement as well since textbooks and tests contain much written text.

Learning style, teaching style mismatch: If a child is oriented toward group experience and teaching emphasizes individual achievement and competition, or if the child learns primarily through physical experience and teaching is primarily verbal and symbolic, achievement may suffer.

Diagnostic Procedures

When attempting to determine the causes for a child's difficulties in mathematics during individualized testing, it is important to follow a few guidelines:

1. Be accepting (not judgmental) of the child. Let the pupil know that you accept him as a person, even when he makes mistakes.
2. Avoid teaching while testing—you are trying to learn what the child knows. If you teach or give hints while testing, your results will be unclear.
3. Do not rush or interrupt the child. Many children do poorly on standardized tests because they cannot function under time pressure. Individualized diagnostic testing is the opportunity for the child to show what he knows without time pressure.

4. Record your results as you test, especially when using teacher-made items. If you do not, you will not be able to remember the details you need to help you determine the child's exact problem.
5. Observe and test for nonacademic causes for the child's difficulties—dyslexia, impulsiveness, hearing problems, reading difficulties, inability to concentrate, and so forth.

Once you have obtained the information you need, plan an instructional program for the child based on your findings. While teaching be continually alert for new information about the child's difficulties. Give short, informal evaluations to determine the student's progress as you teach and modify your techniques and objectives based on these findings.

PLANNING

There are three basic ingredients for running a successful mathematics class—planning, planning, and planning. Many of the discipline problems that face beginning teachers are a result of lack of foresight on their part; that is, they have dead space with nothing planned, they have not anticipated certain problems that students might have, other students are bored because the lesson is too easy, and so on. Of course, a given student might become disruptive for many other possible reasons; however, most of a beginning teacher's problems with discipline can be eliminated if he or she will properly organize instructional time.

Keeping students busy (and out of trouble) is not the only or even the main goal of instruction: their activity must be productive. This book has stressed the importance of teaching children at their level through meaningful activities. Drill should be neither busy work nor over the heads of the learners. Pupils should engage in manipulations, games, and problem-solving activities. If we are teaching students at their level in meaningful ways with a variety of interesting activities, another large portion of discipline problems will be eliminated.

Several considerations are involved in planning and organizing mathematics instruction: planning lessons, presenting lessons, organizing students, and organizing materials. These topics will be discussed in the remainder of this chapter.

Lesson Plans and Objectives

When planning lessons, it is important to have a clear idea of what you wish to accomplish. The best strategy for reaching desired instructional goals is to state them in behavioral terms. What would you like your students to do after you have completed your lesson? "Knowing" something (like when to multiply) is an important objective; however, we need to be able to observe a behavior to infer knowledge.

Some of the actions that students might be asked to do to demonstrate knowledge of mathematics are

Point to
Manipulate
Say
Demonstrate
Divide
Find the square root of
Solve
Place inside

Explain
Remove
Add to
Subtract
Multiply
Write
Compute using a calculator and state results
Measure and state results
Represent
Regroup

When writing objectives in behavioral terms, it is important to be able to state mathematical ideas, numbers, and operations in precise, but sufficiently general terms. For example, if you wish students to be able to do any problem that is similar to 25 + 17 after a unit of instruction, you would not want to have to list all the possible problems from which to choose those that test the child's mastery. Instead, you would describe that type of problem in general terms: "add 2 two-digit whole numbers with regrouping to the tens' place."

When describing problems in general terms, we need to be able to describe the types of values the student is working with. The following is a list of most of the possibilities:

Single-digit whole numbers
Two-digit whole numbers
Three-digit whole numbers, etc.
Integers
Proper fractions
Improper fractions
Mixed numbers
Decimal numerals in tenths (e.g., .3, .7)
Decimal numerals in hundredths (e.g., .23, .05)
Decimal numerals in tens, tenths, etc. (e.g., 2.5, 37.006)
Mixed decimals (e.g., 3.33 1/3)
Percentage numerals
Percentages greater than 100%
Mixed percentage numerals (e.g., 12 1/2%)
Square roots
Repeating decimals
Nonrepeating decimals

In writing objectives it is important to indicate the type of material that will be used. The material could be numeration blocks, chips, pencil and paper, calculator, or computer. We need to specify the type of material used because there is often a great deal of difference in the understanding and skill required to solve a problem with two different kinds of materials. Compare, for example, the different skills needed to find an answer to 127 × 19 using a calculator, or using pencil and paper.

Figure 19.3 contains the major types of materials that might be used to demonstrate mastery of mathematical skills, and the skills themselves. With this chart it is possible to generate most of the objectives relevant to elementary mathematics. To generate objectives, start in the left-hand column and select a material, move to the center column and select a type of numeral, and, finally, move to the

FIGURE 19.3
Materials for teaching mathematics.

The student will

Column 1	Column 2	Column 3
A Represent	a 1-digit numeral a 2-digit numeral a 3-digit numeral a numeral greater than 999 a proper fraction a mixed numeral a decimal numeral a percentage	A with a physical model with a picture by writing the names of the places in expanded notation verbally
B Say Write		B when given [illegible]sical model for the number when shown a pictorial model for the number when given the values in each place of the number when shown an expanded notation sentence when given the numeral verbally

C Add with regrouping Add without regrouping	1-digit number 2-digit number 3-digit number 4-digit number 5-digit number more than 5-digit number	C to	1-digit number 2-digit number 3-digit number 4-digit number 5-digit number more than 5-digit number	using a physical model using a pictorial model using a calculator using a meaningful symbolic process using the standard algorithm
D Subtract with regrouping Subtract without regrouping		D from		
E Multiply		E by		
F Divide without remainder Divide expressing remainder as decimal Divide expressing remainder as remainder		F by		

(*continued*)

FIGURE 19.3
Continued.

G Add **H** Subtract **I** Multiply **J** Divide	a whole number a proper fraction an inproper fraction a mixed numeral a decimal in tenths a decimal in tenths and hundredths a decimal in tenths,hundredths, thousandths a mixed decimal a percent (<100%) a percent (>100%) a mixed percent	**G** to **H** from **I** by **J** by	a whole number a proper fraction with like denominator a proper fraction with unlike denominator an improper fraction a mixed numeral a decimal in tenths a decimal in tenths and hundredths a decimal in tenths,hundredths, thousandths a mixed decimal	using a physical model using a pictorial model using a calculator using a meaningful symbolic process using the standard algorithm

last column to select an operation. The dotted line illustrates the way to generate the objective "say a number greater than 999 when shown a pictorial model for the number." The purpose of this chart is not to generate every possible combination of materials, numerals, and operations, but to help teachers think of different ways of presenting and testing mathematical skills.

You have many sources for behavioral objectives in mathematics. This book contains 20 or so objectives for each instructional area. Most school districts or states have lists of instructional objectives to be accomplished in the various grades. Textbook publishers include behavioral objectives in teachers' manuals. You will find it useful, however, for you as the teacher to be able to elaborate on these and to write more detailed objectives, especially objectives that make use of models, activities, and technology.

Planning Lessons and Instructional Activities

A good lesson plan in mathematics should start with one or more behavioral objectives. These are the guideposts for the lesson—they provide the structure and direction for the activities that are planned and are the basis for evaluating whether the lesson has been successful. Note that each lesson need not be centered around a unique objective or set of objectives. It may take several lessons, even over a period of weeks, for most students to master an objective like "subtract two-digit numbers with regrouping."

The next aspect of a lesson plan is a list of the procedures used in presenting the lesson. The objective(s) of a lesson relate to important ideas to be mastered. However, statements such as "the students will do pages 25 and 26 in the textbook" describe activities they will be doing during mathematics time rather than the skills and concepts learned and should be included in the *procedures* section of a lesson plan. The amount of detail included in this section depends on your experience as a teacher and on your teaching style. Most beginning teachers find that they need to include many details to help keep themselves on target and to assure that they have no gaps or lapses in their presentation. As they gain more experience with a given type of lesson, they need fewer details written in their lesson plans.

When planning instructional activities for a 30 or 40 minute period, we need to vary the activities. Research has shown that instruction is most effective when the teacher spends between 50% and 75% of class time in instruction and the pupils spend the rest of the time doing application or practice activities (Riedesel and Burns, 1973). Some of the different kinds of instructional activities that occur in a typical elementary mathematics class are listed below.

1. The teacher asks children to manipulate objects at their desks while the teacher directs the manipulations, asks questions, and explains.
2. The teacher demonstrates a mathematical concept to the group using large, demonstration manipulative materials.
3. The teacher demonstrates a concept to children using pictures—she draws on the chalkboard, uses a flannel board, uses transparencies on an overhead projector, or shows slides, films, or filmstrips.
4. The teacher discusses a concept and asks questions, leading children to discover a mathematical principle. (See Chapter 2.)
5. The teacher discusses an abstract or symbolic principle or procedure, such as the steps in an algorithm.
6. Students learn or review mathematical concepts and procedures using computers, programmed textbooks, audio tape, television, or film strips.

During the part of the class in which the teacher is not directly instructing students, they can be doing a variety of things:

7. Students work in groups, gathering and analyzing data.
8. Students work in small groups or individually with manipulatives, calculators, and/or computers to solve mathematical problems.
9. Students work individually on discovery activities.
10. Students work individually on practice activities.
11. Students play mathematical games in small groups.
12. Students work in learning stations, at computers, or in the math lab on specific tasks.
13. Students participate in assessment activities using unit tests, small group assessment, computerized tests, standardized tests, and so forth.

Another important aspect of the lesson plan is a list of the materials needed in teaching the lesson. If the teacher has not thought this through and gathered the appropriate materials before embarking on the lesson, she will invariably discover in the middle of the presentation that something is needed. The time to look for rubber bands or an overhead transparency is not in the middle of a presentation. Nothing detracts from a lesson like a teacher fumbling around trying to find materials while attempting to teach something.

The final step in the lesson is to plan for evaluation. How did the lesson go? Were your objectives met? The evaluation of the lesson should be based on the objectives listed at the beginning of the plan. Some of the ways a teacher may determine if objectives have been met are to

1. Observe children's use of manipulative materials as they attempt to solve problems.
2. Ask children how they obtained a specific answer.
3. Ask a child to explain a procedure to another child.
4. Ask a child to find answers to a specific set of exercises or problems (using a calculator, manipulatives, pencil and paper, estimation techniques, etc.).

Figure 19.4 shows a completed lesson plan for a third-grade math class. Note that at the end of the lesson plan is a spot for comments about the lesson. It is important to note difficulties, thoughts for improvement, or areas of the lesson that were especially successful so that the next time you teach the lesson, these may be considered. After the lesson has been completed, file the lesson plan for future use.

FIGURE 19.4
A completed lesson plan.

Teacher Gloria Chin
Class Third Grade
Group Middle
Date Dec. 5, 1987
Time 10:15 – 11:30

Objective: Use a centimeter grid to find the area of a non-rectangular region.

Procedures:

1. Introduce use of centimeter grid to find areas using overhead projector. Show relationship to tiling (previous day). Find areas of 2 rectangular and 2 irregular regions.
2. Distribute materials, students cut shapes from ditto paper.
3. Students find areas of cut-out shapes from grid paper.
4. Meet with group: discuss 3 of student problems (shapes 1,4,5), evaluate.
5. Students work in pairs, find area of hard surface, bottom of soda can.

Materials:
Overhead, transparent cm grid, transparent shape, transparency markers.
Ditto with shapes, scissors, cm grids (100), empty soda cans.

Evaluation:
Most children got acceptable areas for shapes 1, 4, 5.
Jenny and Sean did not get #5.
All got at least 1 hard surface.
Not enough time for soda can.

Long-Range Planning

About a year ago, I was supervising a beginning teacher in an elementary classroom in the Los Angeles area. After visiting him for three weeks and witnessing adequate instruction in numeration and division, I came the fourth week and saw a complete change in direction. He had taken a weekend seminar on computers in the classroom, had received a computer for his classroom and had decided to use math time for instruction on "computer literacy." Being a computer nut myself, I thought that this was not too bad an idea until I returned the next week and found that all his lesson plans for the past 7 sessions involved the history of computers, word processing, and social studies CAI (computer-assisted instruction): he had completely abandoned mathematics.

Although the above teacher's interest in computers and his desire to use them in his classroom were admirable, his deviation from the more important subject of mathematics was not. He had not considered the impact on his students' education of two or more weeks of no mathematics instruction.

Beginning teachers often lose sight of the forest for the trees. They are so concerned about the momentous task of planning lessons in several different subject areas for several achievement groups that they lose sight of the larger curriculum, the things that need to be accomplished during a term, semester, year, or a child's entire schooling.

Every teacher, before a term begins, should sit down and make a list of the things that need to be accomplished for that period of time for each group in his class. This list should be frequently consulted and updated as the term progresses. Plan books, often a requirement of school districts, can serve this purpose—they will usually have the space needed to briefly describe goals in the major subject areas for the different groups you have in each.

Once long-range planning has been accomplished, it should not be forgotten. Check your long-range goals at regular intervals to make sure that you are on target and that you have not been sidetracked by an impulse or by the needs or desires of a small minority of your class.

A stage in between short-range planning and long-range planning is the *unit*. A unit is a series of lesson plans designed to teach a specific topic like addition, numeration, or geometry. Units typically consist of 5 to 15 lessons and include a detailed assessment of the concepts and skills taught. Planning instruction in units has several advantages. One is that chapters in mathematics textbooks are usually about the length of a unit. Another is that units provide a clear focus for instruction for a manageable period of time. And, after two weeks or so on one topic, both teacher and student are usually ready to move on to something new.

Textbooks

Many beginning teachers mistakenly think of the ubiquitous elementary mathematics textbook as the heart and soul of the mathematics curriculum. They start at page 1 at the beginning of the year and progress page by page, going as far as possible during the year. This is a mistake for a variety of reasons, not the least of which is the need to gear instruction toward individual needs. Chapter 1 in your mathematics textbook will contain material mastered by many of your children in previous grades, but too difficult for others of your pupils.

When textbook publishers develop a mathematics series, they want to appeal to the largest possible audience so that they can sell the most books. They want their text to appeal to inner-city schools, rural schools, as well as upper-middle class suburban and private schools. A second-grade text, for example, might include topics usually mastered by the "average" children in first grade as well as materials not mastered by most children until the end of third grade. Thus, you need to pick and choose within a textbook to find math materials to meet the needs of each child.

Another problem with textbooks is that, since they are printed, they emphasize symbolic understandings and manipulations. Many have a strong orientation toward pencil and paper computational skills, but almost ignore problem solving, estimation and mental arithmetic, mathematical understandings, calculator arithmetic, geometry, and applied mathematics—the topics that are relevant to our technological society.

Recognizing these limitations, however, you will find that math textbooks are still your primary resource for teaching mathematics. They contain instructional and practice activities that can be fitted into many, if not most, lessons. The teacher's manuals of the better series have good suggestions for developing understanding of the topics covered through manipulatives. They contain pictorial sequences designed to develop understanding of appropriate topics; excellent problem solving and applied activities; calculator, computer, and estimation activities; and games and other interesting practice activities.

The most important advice related to elementary math textbooks is that they should be considered resources from which appropriate activities may be selected to fit your mathematics program. Base your mathematics program on your school, district, or state mathematics guidelines and on the needs of your pupils rather than on the sequence of topics in the elementary math textbooks handed to you. Pick and choose activities and materials from a variety of sources, including textbooks when appropriate.

CLASSROOM MANAGEMENT

Not only are teachers expected to manage curriculum, but we must manage students and materials. We must make sure that children get to their various daily activities on time and that they are studying and learning efficiently and effectively. In an active mathematics classroom, we have many materials to manage as well—base-ten blocks, pattern blocks, calculators, and computers, to name a few.

Organizing Students

Probably the most common way of organizing students in mathematics is to treat the class as a unit, teaching the same concepts and skills to all students regardless of their background. This practice, while the easiest to manage, is not the best because it meets the needs of only a small part of the class. Even in kindergarten there is a wide range of skill and knowledge among students—some can count past 100, read single-digit numerals, and do mental addition and subtraction while others cannot count to ten and cannot make sets to show the meaning of single-digit numbers like five. As children grow older, differences between them become even more exaggerated; a typical seventh grade math class may have children ranging in achievement levels from the second grade to the tenth grade.

Grouping

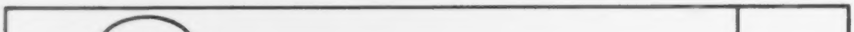
Several methods for organizing students besides the single group method are possible. One technique is to divide the class up into groups according to their mastery of objectives. In this method, the teacher gives a diagnostic test (a test that provides a detailed profile of the student, listing skills mastered and not mastered) at the beginning of the year and divides the class into from two to four groups based on the skills that need to be mastered by the students. At the end of each unit, the teacher gives criterion tests to determine if objectives have been mastered. On the basis of these tests, she may form new groups. Usually, however, most children stay with more or less the same group of children throughout the year.

When grouping children for mathematics instruction, your planning needs to be even more rigorous than when using a single-group plan. You will need to plan activities for each group for the entire class period. That means two to four lesson plans. While you are presenting new material to one group, the other groups are working in the math lab, on the computer, with manipulatives, playing math games, doing pencil and paper drill, and so forth.

Figure 19.5 shows a possible layout of a class with three math groups. It is the beginning of the period and the teacher is presenting the lesson to the lowest group, the group least likely to remember instructions from the previous day and least likely to be able to follow written instructions. Group 2 is doing a variety of problem-solving activities and is scattered around the room. Group 3 is using geoboards and doing geometry activities presented on task cards. Of course, this arrangement will vary from day to day and even within a given math period.

You may ask how a teacher manages to teach three or four groups at a time. The answer, again, is organization. It is also helpful to have an aide, parent assis-

FIGURE 19.5
A three-group layout.



tants who can work with individual students, peer tutors, or several computers and good instructional software.

One way that many teachers avoid the extra planning needed to manage multiple groups in a single classroom is to group across classrooms. Here, several teachers at a grade level cooperate, scheduling their math periods at the same time, grouping children from all the classes according to achievement, then teaching one of those groups. In a group of three or so teachers, one takes the "low" group, another the middle group, and another the high group. This type of grouping requires a high degree of cooperation, planning, and philosophical agreement among teachers.

A final note about grouping: it is a good idea to totally rearrange your groups on occasion, bringing together the diverse members of the class for a unit or several class periods. A unit on geometry, for example, will require a different set of prerequisites from other topics covered, allowing for a different mix of students in groups. You may even wish to allow students to choose their group partners for a special type of activity like an outdoor problem-solving activity or an integrated math-science or math-art project.

Individualized Instruction

Ideally, every child should be taught exactly what he needs at his level using the type of learning experiences that relate to his learning style. Some teachers are able to accomplish this to a certain degree. In an individualized classroom, every child is working at his own pace on the objectives he has not yet mastered. In this type of classroom, the teacher is rarely able to present lessons to children. Instead he manages instruction, tracking students' progress, distributing and checking tests, copying, arranging and dispensing instructional materials, answering students' questions, and delivering very short bits of instruction, and rewarding students' progress.

In this mode of instruction, the teacher often sets forth the curriculum as a sequence of pages from a text to be read, computer programs to be run, filmstrips and audio tapes to be absorbed, worksheets to be completed and games and other activities to be done. These will be listed on charts. Each time a student completes a required activity, a star or other mark is placed in the appropriate place on the chart. When a student has completed a required number of tasks in a unit, he will be assessed. If he passes the assessment at the desired level (typically, 80%), he will move to the next unit. If not, he will work on the material some more until he feels that he is ready to retake the criterion test. A student never moves out of one unit to the next until he has mastered the current unit.

A completely individualized system of instruction has several advantages. First, each student's needs are met if the system is properly managed. Second, certain students thrive in an individualized approach to learning—some students will progress two or three grade levels during a single year when they learn at their own pace.

However, several problems are related to a completely individualized instructional approach. Such a method of teaching involves a great deal of work, especially in the initial stages. In addition, many teachers miss the satisfaction of presenting lessons to groups of children. Very few good materials are available for individualized instruction—math textbooks are of little help and "programmed," self-paced materials tend to emphasize rote, symbolic skills. Even existing computerized instruction is inadequate for anything more than a few hours of individualized instruction. Finally, and perhaps most importantly, many children who lack self-discipline and/or self-confidence do not do well in an individualized mode of learning because they need the continual presence of a teacher to motivate them and/or they need to be able to work cooperatively with other children.

One answer to the above-mentioned problems with individualized instruction is to combine grouping and individualization. Make it possible for your gifted and highly self-motivated students to work at their own pace and divide the rest of the class into two groups. This way, you may be doing an even better job of meeting every child's needs in your class than if every child had his or her own prescribed plan of learning.

Managing Materials

When you become responsible for your class, the only materials you have in your classroom may be books, paper, and a few games and activities developed in your methods courses. As time goes on, however, you will find that you quickly gather an abundance of materials. This is especially true if you follow the advice of this book and teach mathematics through activities.

One way to manage materials is to gather a large number of shoe boxes. Laminate them with brightly colored paper and mark each box with a large number and a brief description, as illustrated in Figure 19.6. Each box should be completely self-contained with materials, instructions, and possibly, answers. The boxes may contain games, small sets of manipulative materials, problem-solving activities, worksheets, unit tests, quizzes, construction materials, and so forth. You may keep larger sets of manipulative materials in large plastic bins or trays.

Place the boxes in order on easily-accessible shelves (see Figure 19.7). These boxes may be referred to by number on assignment sheets, individualized programs, or task cards. Note: make it the students' responsibility to replace all materials used.

FIGURE 19.6
A math box.

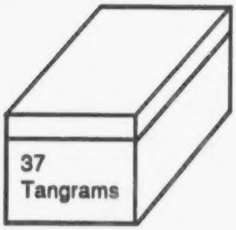
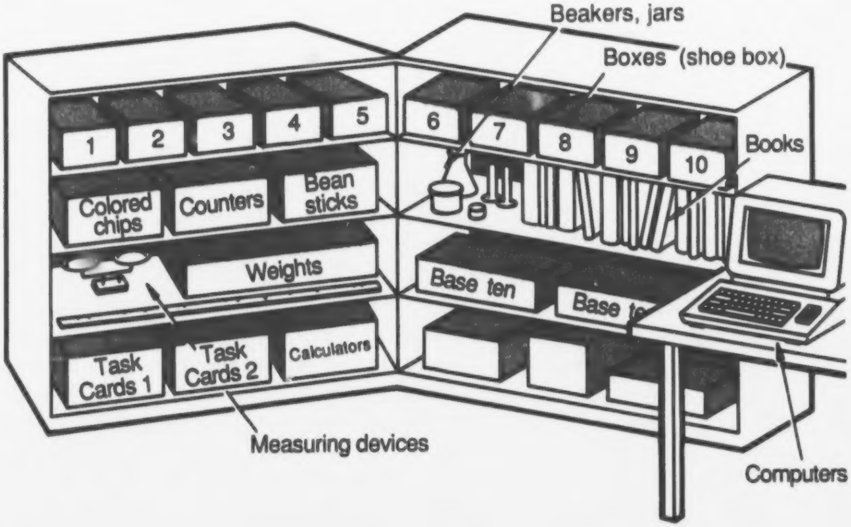


FIGURE 19.7
A math lab area, containing boxes and trays.



Another method for managing materials is to use file folders for all worksheets, tests, puzzles, etc. and to use trays for sets of manipulative materials. File folders should be numbered and labeled and referred to by number on assignment sheet.

If you have many materials, or if you run a math lab or a math learning center, you may wish to computerize your inventory. One possibility, especially if your inventory is very large, is to use a data base program, such as AppleWorks (data base program) PFS File, or Lotus 1-2-3. You may wish to set up your fields as shown in Figure 19.8 below.

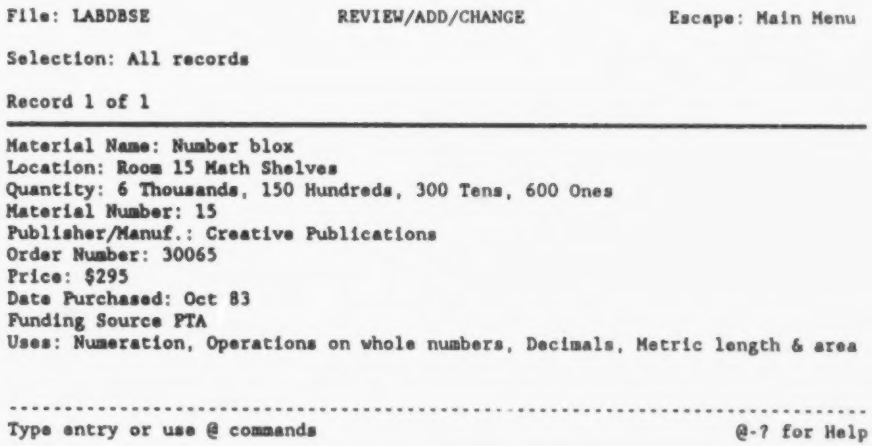


FIGURE 19.8
A data base of math lab materials.

```
File: LABDBSE                 REVIEW/ADD/CHANGE                 Escape: Main Menu

Selection: All records

Record 1 of 1
______________________________________________________________________________
Material Name: Number blox
Location: Room 15 Math Shelves
Quantity: 6 Thousands, 150 Hundreds, 300 Tens, 600 Ones
Material Number: 15
Publisher/Manuf.: Creative Publications
Order Number: 30065
Price: $295
Date Purchased: Oct 83
Funding Source PTA
Uses: Numeration, Operations on whole numbers, Decimals, Metric length & area

------------------------------------------------------------------------------
Type entry or use @ commands                                       @-? for Help
```

In case you are not yet sure as to the types of things that should be included in a mathematics laboratory, a list is provided in Table 19.1. Of course, each teacher has his or her preferred materials that may or may not be included here. The materials included in your lab will also depend on the grade you are teaching. Table 19.2 might be considered a wish list for a fully stocked lab for the middle and upper elementary grades.

Distributing Materials

One of the biggest problems beginning teachers have with manipulative materials is distributing them to an entire class or group efficiently and quickly. I have seen teachers waste ten minutes gathering and distributing the materials the children will be using in a math lesson. Often by the time the children finally have the appropriate materials, the class is out of hand and it takes another five minutes to calm them down and begin the lesson.

The worst way to distribute sets of manipulative materials is to distribute all the materials yourself without any prior planning. Invariably, if you do not check the presence and location of the materials before the lesson is to begin, you will find some pieces are missing or misplaced. Also, it takes time to distribute sets of manipulatives yourself. This is especially true if the materials are not properly ar-

TABLE 19.1 List of materials for math lab.

Quantity	Description
30 sets	Bundled sticks (50 sticks and rubber bands per set)
	A variety of counters (beads, dried beans, etc.)
30	Loops of yarn
30 sets	Colored chips (20 white, 20 blue, 10 red, 10 green)
1	Math fun box (see Chapter 6)
30 sets	Numeration blocks or bean sticks
30	Place value charts and markers (e.g., dried beans)
30 sets	Fraction Tiles, Fraction Bars (TM) or teacher made fraction materials
30	Decimal place-value charts and markers
2	Fraction table (see Chapter 12) and paper strips
30 sets	Percent elastics and/or transparencies (Ch. 14)
30 sets	Play money
5	Clocks
30	Meter sticks
30	Centimeter rulers (30 cm. long)
5 sets	Solid geometric shapes (cone, prism, etc.)
5 sets	Metric graduated cylinders
5	Pan balances with metric weights
30	Geoboards and rubber bands
10	Miras (TM) (See Chapter 16)
1 set	Calculators
	A variety of math games in shoeboxes (Ch. 19, 20)

ranged—imagine counting out 20 units, 15 longs, and 5 flats for 15 children. It would be better to have each child's set in its own container and all containers in a box so that you can quickly walk around the room and distribute them. Better yet, select a child for each group of five children and have them get enough sets for their group.

The following are some suggestions for distributing manipulative materials to a large group:

1. Plan ahead. Before school begins or during a break, make sure that all materials necessary are where they should be, in a readily accessible place. Group the materials so that they can be distributed easily. Place individual sets of materials in plastic containers, large envelopes, or plastic bags.
2. Have selected pupils help you distribute and collect the materials.
3. If your materials tend to disappear, stack or arrange them in their container in an easily-recognizable way so that you can verify at a glance that all the materials have been returned.
4. Return all sets of materials to the proper location in the room so that they will be available for their next use.

A final note about classroom management: do not expect to become an expert disciplinarian, presenter of lessons, diagnostician, and manager overnight. It usually takes a few years to get the whole operation running smoothly, and if you are a good, innovative teacher, about the time you do, you will decide to implement a new system.

SUGGESTED READINGS

Ashlock, R.B. *Error Patterns in Computation* (3d ed.). Columbus, OH: Merrill, 1983.

Fennell, F. Mainstreaming and the mathematics classroom. *Arithmetic Teacher* 32 (3), November 1984: 22-27.

Larson, C.N. Organizing for mathematics instruction. *Arithmetic Teacher* 31 (1), September 1983: 16-20.

Underhill, B., Uprichard, E., and Heddens, J. *Diagnosing Mathematical Difficulties*, Colombus, OH: Merrill, 1980.

QUESTIONS

1. Using the form in Figure 19.3, design a lesson plan for teaching regrouping in addition. Include concrete and symbolic activities.
2. As in #1, design plans for a lesson on networks.
3. Discuss a management system for computer software.
4. Go to the curriculum center in your school and find out what kind of mathematics diagnostic instruments are available. How do they compare? Which could be used for in-depth diagnosis?
5. Administer *Key Math* to an elementary pupil. Based on the profile obtained, plan specific diagnosis using the behavioral objectives in this book and the diagnostic items in Appendix A.
6. Design a pencil-and-paper diagnostic test to measure a student's ability to add two-digit to one-digit numbers (vertical form) with regrouping.
7. Identify the error pattern below:

$$\begin{array}{r} 180 \\ -\ 46 \\ \hline 146 \end{array} \qquad \begin{array}{r} 206 \\ -\ 122 \\ \hline 124 \end{array} \qquad \begin{array}{r} 500 \\ -\ 341 \\ \hline 241 \end{array}$$

C H A P T E R

20

Practice Activities & Games in Mathematics

Practice is an important part of any learning experience. In learning mathematics, children need to gain proficiency with mathematical procedures, algorithms, "facts," estimation and mental computation techniques, problem-solving strategies, and the use of calculators and computers for finding answers in computational situations. They should not need to get out a calculator to determine how much each of three persons owes at a McDonalds if the bill is $9 and it is being split equally or what their change should be if their bill is $7.50 and they give the cashier a $10 bill, but should be able to use one to add long columns of multidigit figures or to do long division.

Chapter 2 introduces some basic principles related to practice in mathematical learning: drill should follow understanding, practice perfects what is practiced, and practice can be enjoyable. The mathematical principles behind computational procedures should be presented by the teacher and discussed as a group, and examples should be attempted by individuals with large or small group feedback before individual practice is begun. Thus, before the student is isolated for 15 or 20 minutes of drill, he has had sufficient opportunity to understand *why* the procedure works, *how* it works, and *how* and *when* it is used and has either physically or mentally gone through the steps and has received feedback on the procedure he will practice.

When planning drill for students, use a variety of procedures. Since children have different learning styles, some may profit more from drill in a game setting and others from no-frills worksheets. Most children will benefit from variety, the spice of a lively mathematics class. We can explore many possibilities for the practice of mathematics.

INDIVIDUAL STUDY AND TIMES QUIZZES

Study periods followed by timed drills can be used in the lower grades to assist in memorization of the "facts"—addition, subtraction, multiplication, and division with single-digit addends, differences, factors, or quotients, respectively. The purpose is for students to develop immediate recall of the answers to the arithmetic facts. For example, when the child sees 3 × 8 = , she immediately states or writes 24.

In this technique, after the teacher has spent sufficient time developing understanding of the concepts involved and the student has developed appropriate strategies for computing answers (e.g., counting techniques for addition facts or repeated addition for multiplication facts, so that if she cannot recall the answer she will be able to calculate it), she is given a study list similar to a spelling list with the facts that have not yet been mastered. The student looks at the facts (e.g., 8 + 7 = 15) one after another, turns the paper over and mentally pictures each and, perhaps, states it subvocally. The student then writes the facts and checks to see if they are right. In order to encourage immediate recall, the teacher gives timed quizzes where a student has, for example, 10 problems without answers to complete in 30 seconds. You may want to put a mixture of problems on the quiz—some the student has mastered and some on which he is currently working. If you are sufficiently well organized, the entire class can participate in the timed drill activity even though they are each working on different facts. You may make three or four sets of quizzes and distribute them to appropriate students. Five minutes of study a day on a limited number of facts (perhaps 5) would be appropriate for most children.

GROUP STUDY OF "FACTS"

The basic facts may be studied in pairs or small groups using flash cards with such problems as 5 + 6 on one side and the answer or complete number sentence on the other. Many children, especially those from certain minority cultures, may prefer cooperative, group study to individualized study. For example, you may have a group of four students who need to work on 6 × 8, 7 × 8, 8 × 8, and 9 × 8. Each student takes a card and silently tries to memorize that fact. They rotate the cards clockwise and continue their study. Finally, one student takes all four cards, says the name of one of the other students, and holds the flash card for the student to see. The student must then state the answer immediately. If the answer is wrong, the leader shows the student the other side of the card. This continues until all children have given several answers. If some of the students are still unsure of some of the facts, there is more study and quizzing.

GAMES

One of the best ways to practice computational procedures is through group games. A number of curriculum companies sell instructional games for use in the elementary mathematics classroom. As with instructional software, games should be evaluated to determine if they have educational value. The following is a list of considerations when evaluating or designing games for use in the mathematics classroom.

1. Students' participation and success in the game should involve correct solutions to mathematical questions. For example, a student might have to correctly solve a two-digit by one-digit multiplication problem in order to advance his token in a board game. If he gives the wrong answer, he stays where he is or moves backward.

2. The mathematical exercises around which the game is designed should relate to a specific mathematical objective and this objective should correspond to your current instructional objective. For example, a game might be designed to practice associating a written fraction with the picture of a fraction (refer back to Activity 12.1 in Chapter 12). In order for students to benefit from this game, it must be used at the correct point in the curriculum. The fractions game in Activity 12.1, used with students who have already mastered fractions, would be a waste of time no matter how entertaining it is. You should use games that reinforce a particular concept or operation at about the same point you would use pencil and paper drill over that same topic.

3. Consider *time on task* of students playing the game. Time on task can be defined as the portion of the time the game is being played that an individual student is actually doing math problems. For example the time on task in a board game played for 30 minutes by five children where each child in turn takes a problem card, solves it, then takes appropriate action in the game would be about .2 × 30 or 6 minutes (only one out of the five players is actually solving math problems at any given time in the game). On the other hand, in a math bingo game played for 30 minutes, the teacher shows the class a math problem, the entire class works the problem, then checks their bingo card to see if the answer is there. Here the time on task is nearly 30 minutes for each student since virtually all children involved in the bingo game are working math problems during the entire time the game is played. Games with low time on task should be avoided because they make poor use of class time, no matter how motivational they are.

4. Instructional games should have a balance between chance and skill. Games like chess which are based entirely on skill are not motivational to poor players while dice games, based entirely on chance, do not reward skill. The presence of chance allows players with modest skills to compete with highly-skilled players. It also keeps the game interesting until the end since players lose interest in contests where the losing player has no chance to catch up and win. Some ways to include chance in a game are through dice, random selection of cards, playing boards with pitfalls, or use of spinners.

5. Instructional games, to be of value to students, should provide immediate feedback. Feedback might come from the game materials themselves (on the back of cards or on a check sheet), from calculators, or from others playing the game. Students should know quickly if they answered the question correctly and, if not, why not. The result of failure to solve the problem correctly should be no advancement toward the objective of the game.

6. The rules of the game should be simple. A rule of thumb: It should take no more than a couple of minutes for a child to master the rules of the game and once mastered, the child should be able to teach another child to play it.

The checklist in Figure 20.1 uses the above criteria for evaluating mathematics games.

FIGURE 20.1
A game evaluation form.

Name of Game ______________________

Publisher of Game ______________________

Price ______________________

Number of Participants ______________________

Materials ______________________

Mathematical Objective ______________________

Evaluation:
1. Does success in game depend on correct solution to math problems?
   Yes No

2. Are math exercises in game related to its stated objective?
   Yes No

3. Estimated proportion time on task:

4. Balance between chance and skill:
   Mostly Skill Evenly Balanced Mostly Chance

5. Speed of feedback:
   Immediate Somewhat Delayed Long Delay None
   How is feedback provided?

6. Simplicity of rules:
   Very simple Simple Somewhat complex Very complex

DESIGNING YOUR OWN GAMES

Almost any commercial game can be used as a model for designing a classroom game or as a part of a practice activity. For example, math exercises could be placed on small cards, students could take turns drawing the cards from the deck and working the problems listed. They would check the answer and then, if the answer is correct, take their turn at whatever game they were playing, such as Race Track. In this section, we present several models for games that can be adapted to practice any algorithm in arithmetic. We discuss bingo-type games, board games, card games, and puzzles.

Bingo-Type Games

Materials:

1. *Bingo cards.* Draw and reproduce cards containing numbers or answers from the problems the children are expected to solve when playing the game, as illustrated in Figure 20.2. Each bingo card should be different from all others.

FIGURE 20.2
A CHINGO card.

CHINGO

40	16	257	372	19
80	60	15	18	110
100	17	FREE SQUARE	20	200
217	70	14	120	251
30	90	147	203	452

In order to keep students' attention, the participants should be able to find answers on their cards to a half to a third of the problems worked. Thus, if you are using 5 by 5 bingo cards, there should be between 48 and 72 problems to work from, each producing a different answer.

Another way to handle bingo cards is to make a worksheet with all possible answers randomly arranged on a grid, as in Figure 20.3. To play, students mark a 4 by 4 or 5 by 5 square of numbers which becomes their bingo card. This is a real time saver for the teacher: you only need to make one worksheet which serves as the playing board (bingo cards) for all players.

2. *Chips.* Make chips with problem (or problem numbers) and answers. These may be half-inch squares cut from tagboard or plastic chips with stick-on labels with a problem number in one color on one side and the answer to that problem on the other.
3. *Materials to represent problems.* There are numerous possibilities for delivering problems for students to solve. For younger children, you may read them the problem, write it on the board, or represent it with materials on the overhead. With older students, you may use a worksheet with numbered problems as in Figure 20.4.
4. *Markers.* With re-usable bingo cards, you will need bingo markers: squares of colored paper, dried beans, or colored chips.

Procedures: Draw a problem chip from a box and read the problem number, if you are using a worksheet. Students look for the problem on their copy of the worksheet, then find an answer to the problem, look for the obtained answer on their bingo card, and, if they find it, they mark it. The teacher places the chips in front of him with the answers up in ascending order so that when a student has a "bingo," the answers in the bingo can be read and the teacher can easily check to see if the student's answers correspond to the correct answers. For young children the process is similar, except that the teacher states the problem, represents it with materials, or writes it on the blackboard instead of using the worksheet. You may

FIGURE 20.3
A bingo playing card.

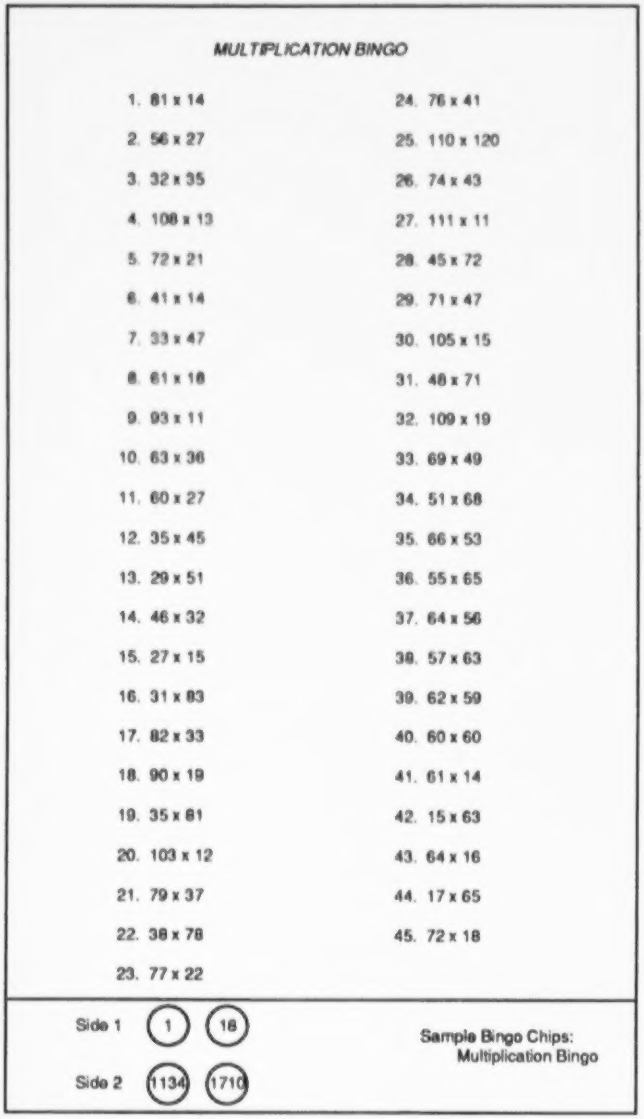
1512	1404	3116	1551	1023	2268	1575	1472	2573
1710	1236	2964	3116	1134	1120	3116	574	1098
1620	1479	405	2706	2835	2923	1694	405	1575
1134	1404	1551	2268	1479	2573	2835	2964	1512
1098	1620	1472	2706	1236	1694	1120	574	1023
1575	405	1710	2923	3116	1134	1023	1479	2706
2923	3116	1512	574	2268	1472	1710	2964	120
1551	1620	405	2835	1694	3116	3182	1694	2964
2923	1236	2835	1710	2706	2573	405	1472	1479
1575	1620	2268	1023	1098	1551	574	1404	1120
1512	1134	1575	1236	2706	3182	1512	1551	1575
1551	3116	1098	1023	2835	1236	1694	1575	1134

wish to accept several bingos before asking children to clear their bingo cards for a new game. Finally, you may wish to give small prizes to enhance the interest in the game.

Discussion: Advantages of mathematics bingo are that this type of game is motivational, time on task is high, and it is easily adapted to nearly any math skill. The disadvantage is that feedback is not immediate except for those who score bingos. One way to increase feedback is, after every second problem and after sufficient time has passed for everyone to work the problem and search for the answer on their card, to tell the students the correct answer.

Board Games

Board games are among the most flexible of practice tools for the classroom. Typically, a board game consists of a playing surface with a path that players traverse with their markers. The paths may be an endless variety of shapes or mazes with pitfalls, detours, "chutes and ladders," alligator ponds, or any other devices you can imagine. Some commercial examples are the circular path with spokes (Trivial Pursuit), a looping path (Monopoly), a zig-zag path from one corner of the board to the opposite corner (Chutes and Ladders), or a simple straight path (Racetrack). They also consist of a method for determining the number of spaces to move (e.g.,



MULTIPLICATION BINGO

1. 81 x 14
2. 56 x 27
3. 32 x 35
4. 108 x 13
5. 72 x 21
6. 41 x 14
7. 33 x 47
8. 61 x 18
9. 93 x 11
10. 63 x 36
11. 60 x 27
12. 35 x 45
13. 29 x 51
14. 46 x 32
15. 27 x 15
16. 31 x 83
17. 82 x 33
18. 90 x 19
19. 35 x 81
20. 103 x 12
21. 79 x 37
22. 38 x 78
23. 77 x 22
24. 76 x 41
25. 110 x 120
26. 74 x 43
27. 111 x 11
28. 45 x 72
29. 71 x 47
30. 105 x 15
31. 48 x 71
32. 109 x 19
33. 69 x 49
34. 51 x 68
35. 66 x 53
36. 55 x 65
37. 64 x 56
38. 57 x 63
39. 62 x 59
40. 60 x 60
41. 61 x 14
42. 15 x 63
43. 64 x 16
44. 17 x 65
45. 72 x 18

Side 1 ① ⑱

Side 2 (1134) (1710)

Sample Bingo Chips:
Multiplication Bingo

FIGURE 20.4
A multiplication bingo worksheet.

dice or spinner). Often, they also include a stack of cards that give added instructions for penalties or rewards (as in Monopoly), or contain questions (as in Trivial Pursuit). The educational version of a game will usually include a stack of cards with questions that students must answer to advance in the game. This is what gives board games their flexibility—with one playing board and accompanying set of rules, countless skills (mathematical and other subjects) may be practiced by simply developing new sets of playing cards.

Board games discussed here are either (a) linear path, or (b) continuous-loop path.

a. Linear paths progress from a starting location to a finish. The object of the game is to be the first to reach the finish area. Figure 20.5 shows an example of a linear path with several detours.

b. In a continous-loop path, the major objective of the game is to amass a commodity like money (Monopoly), or some type of token (the wedges that show that you have answered the question in a category in Trivial Pursuit), and the amount of territory covered during the game is irrelevant. In most continuous-loop programs, the player may be anywhere on the loop when he wins. In others, the goal is to reach "home"; however, a player may not win until the proper number of tokens has been obtained. As with the linear path-type board games, players usually roll dice or spin spinners to indicate how far they move. The drill aspect of the game is introduced through a stack of cards containing problems, or numbers on the spots where players' tokens land indicating a problem to be worked. Figure 20.6 shows a continuous-loop game board.

Example 1: Linear Path Board Game

Materials:

1. Playing board (see Figure 20.5).
2. A die.
3. Problem cards.
4. Tokens (e.g., little plastic toys).
5. An answer sheet or some other way to determine if the answer is correct.

Procedure:

1. Between two and four students may play at one time.
2. Players throw the die to see who starts (the highest number starts; if there is a tie, the two people who tie throw again).
3. The first player draws a card and attempts to answer the question. If the answer is correct, he throws the die and moves his token the number of spaces indicated.
4. Play moves counter-clockwise until someone reaches the Finish area.

Example 2: Continuous-Loop Mathematics Game

Materials:

1. Playing Board (see Figure 20.6).
2. Red, white, and blue chips.
3. Set of problem cards.
4. Tokens for each player.
5. A die.

Procedures: The procedures for playing this game are printed on the game board in Figure 20.6.

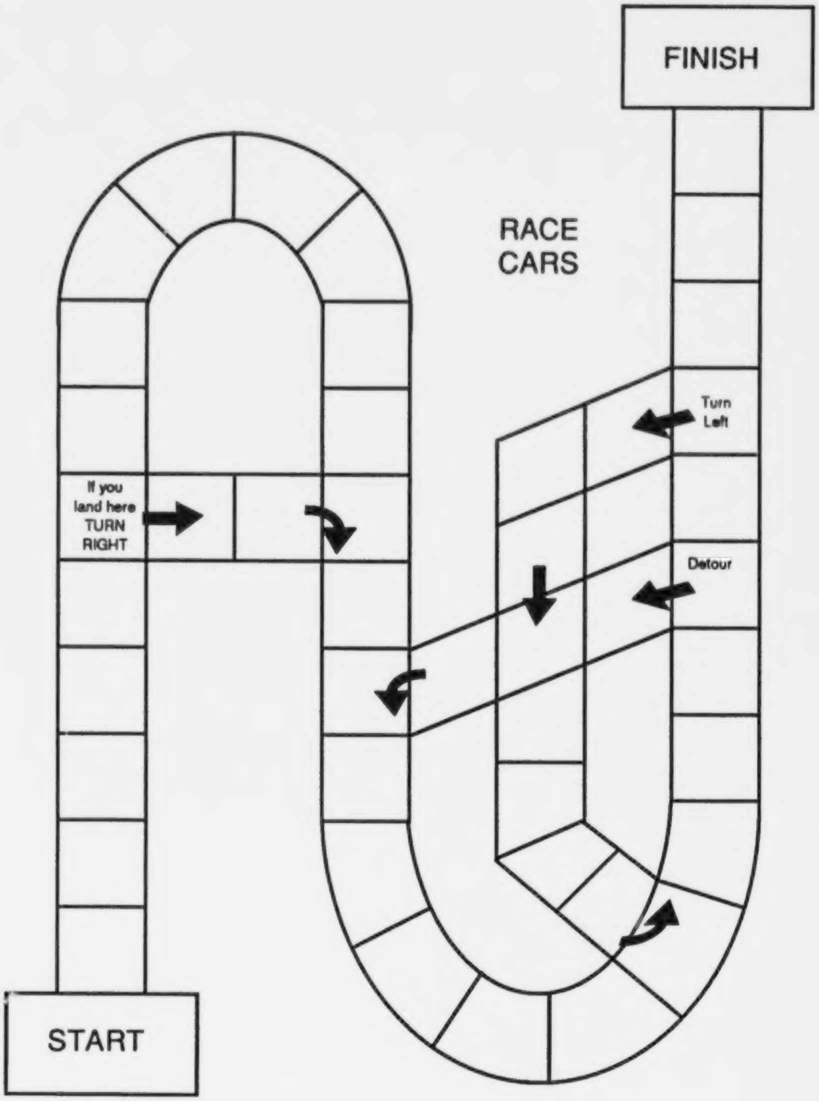
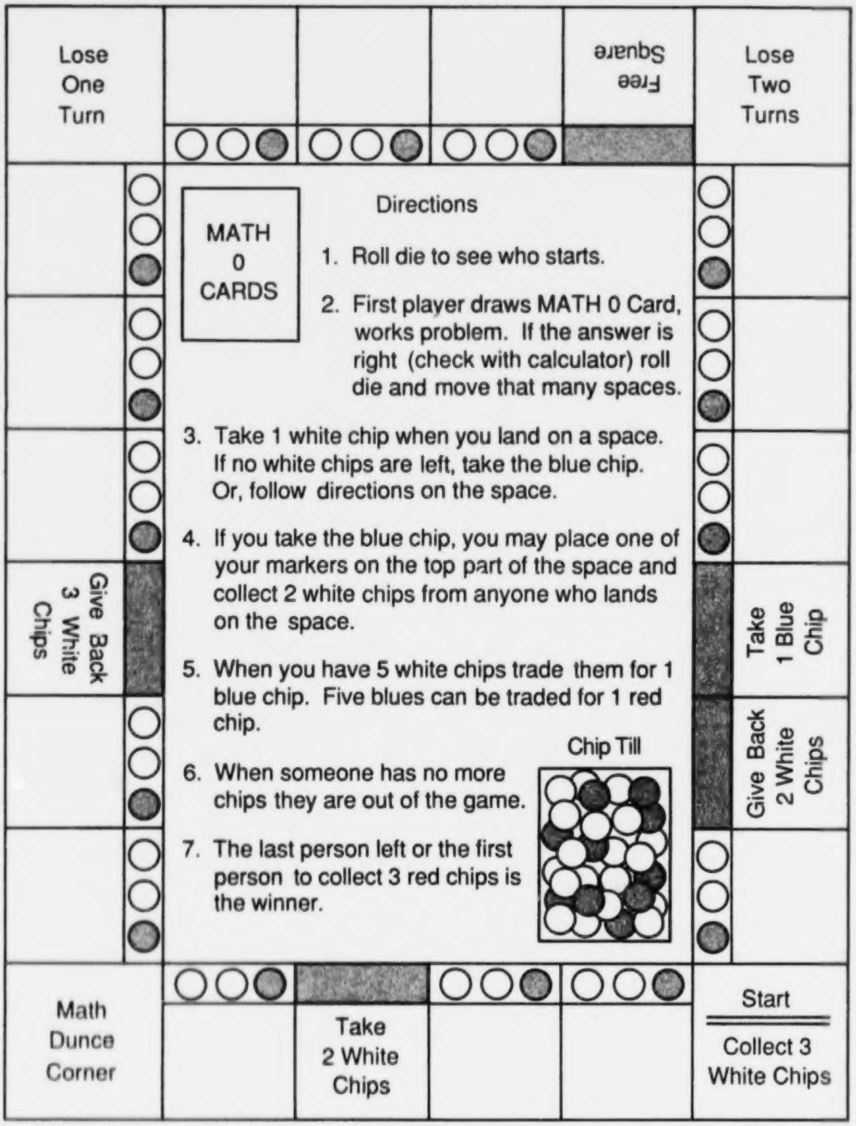


FIGURE 20.5
A linear path game board.

Card Games

Card games have been popular for many centuries in the Western world and the number of games played with cards is staggering. Most of these, however, are of little value in meeting specific educational objectives. Usually, the rules of play of specific card games are more complex than any arithmetic skills that would be practiced in the game. Bridge, for example, uses mathematics for keeping score, for determining face values of cards, and in determining how to bid, but the cognitive skills needed to understand and master the game would typically be obtained long after a child would master the mathematical skills used in the game.

FIGURE 20.6
A continuous-loop game board.


Lose One Turn
Free Square
Lose Two Turns
MATH 0 CARDS
Give Back 3 White Chips
Take 1 Blue Chip
Give Back 2 White Chips
Chip Till
Math Dunce Corner
Take 2 White Chips
Start
Collect 3 White Chips

Directions

1. Roll die to see who starts.
2. First player draws MATH 0 Card, works problem. If the answer is right (check with calculator) roll die and move that many spaces.
3. Take 1 white chip when you land on a space. If no white chips are left, take the blue chip. Or, follow directions on the space.
4. If you take the blue chip, you may place one of your markers on the top part of the space and collect 2 white chips from anyone who lands on the space.
5. When you have 5 white chips trade them for 1 blue chip. Five blues can be traded for 1 red chip.
6. When someone has no more chips they are out of the game.
7. The last person left or the first person to collect 3 red chips is the winner.

However, some card games such as Old Maid have rules that are simple enough to make them of value in mathematical practice. An example of a card game adapted for math practice is Kiss the Spider, a variation of Old Maid.

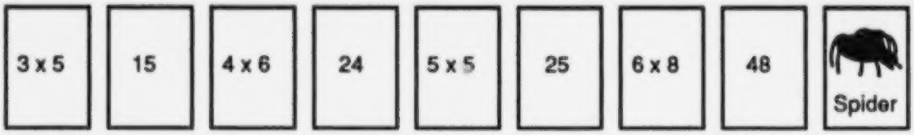
Example:

Objective: Associate a single-digit multiplication problem with the correct answer.

Materials: (See Figure 20.7.)

1. Set of cards marked with the multiplication "facts."
2. Another set with the answers to each "fact."
3. A spider card.

FIGURE 20.7
Spider game cards.



and so on.

Procedure: Deal all the cards to the players (3 - 6). Each player removes all *books* (problem-answer combinations) from his hand and places them in front of himself. Players take turns in a clockwise direction removing one card from the person's hand on their right, without looking at the front of the cards. Whenever this results in a problem-answer book, the player removes it from his hand and places it with his other books. When one person is out of cards, the game is over and the person with the most books is the winner. The player stuck with the spider card must kiss the spider.

Puzzles

Puzzles are perhaps the most popular game-type activity for drill and practice in the regular classroom because they can be completed individually by students at their desks. They can thus be fit into any type of classroom management scheme (large groups, small groups, individualized program, or diagnostic teaching). Since instructional puzzles are designed so that the obtained answers fit into some sort of pattern, they are usually self-checking, thus providing immediate feedback.

Coloring Puzzles

This type of puzzle usually consists of a picture divided into regions. The regions contain either (a) a math exercise, (b) a problem number, or (c) a problem answer.

a. If the regions contain problems, the worksheet should contain a key that tells the student what number to color the parts of the puzzle, based on the answer obtained. For example, *if the answer is between 3 and 4, color the region blue.*

b. If the regions of the puzzle are small or the problems to be solved are long, corresponding numbers may be placed in the regions. Again, the answer to the problem determines the color of the region, or whether or not it should be darkened in with a pencil, or marked some other way.

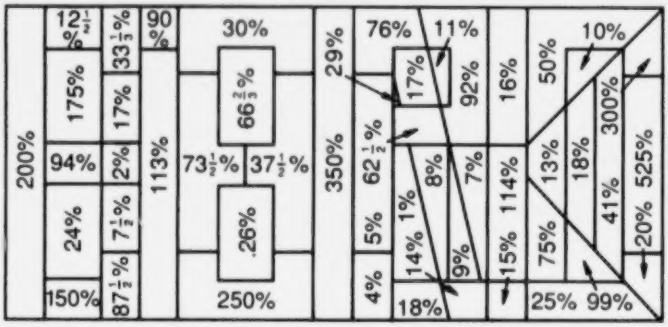
c. Figure 20.8 shows an example of a coloring puzzle where the regions contain answers rather than problems or problem numbers. In this puzzle, the student works the problems, looks for the answers in the puzzle, then colors in each region containing an answer. When she is finished, the colored regions together form a picture or message. In Figure 20.8, the message is "C3P0."

Connecting Dots

In this type of puzzle, the student must work a mathematics problem, then look for the answer next to a dot on the puzzle. The pupil connects the dots in sequence, starting with the answer to the first question, drawing a line to the answer to the second, and so forth. As with other puzzles, this type of puzzle is self-checking, or at least it is checkable at a glance by the teacher. On the other hand, if the student makes many errors, the picture will not materialize, nor will it be easy to determine

FIGURE 20.8
A coloring puzzle.

Name: ____________________


200%
12½%
175%
94%
24%
150%
33⅓%
17%
2%
7½%
87½%
90%
113%
30%
66⅔%
73½%
37½%
.26%
250%
29%
350%
76%
11%
17%
92%
62½%
8%
7%
5%
1%
14%
9%
4%
18%
16%
114%
15%
50%
10%
13%
18%
300%
41%
75%
525%
20%
25%
99%

Directions: Convert the fractions below to percentages. Look for the answer in the design and shade in the area containing the answer. When you are finished, a message will appear.

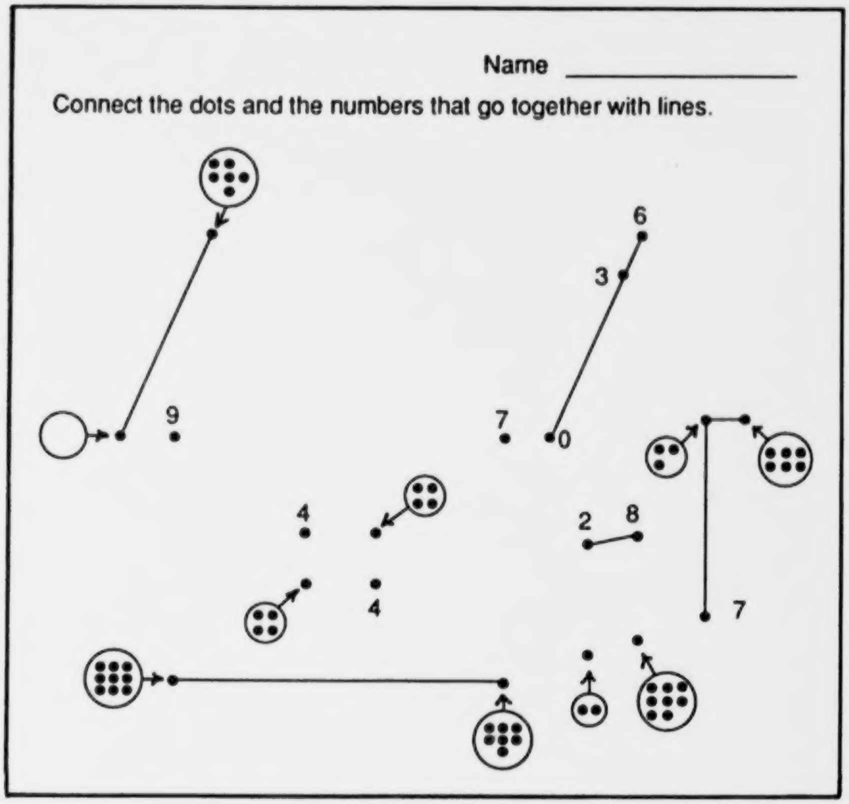
1. $\frac{1}{2}$ = __________%
2. $\frac{1}{8}$ = __________%
3. 2 = __________%
4. $\frac{13}{100}$ = __________%
5. $\frac{3}{4}$ = __________%
6. $\frac{2}{10}$ = __________%
7. $\frac{1}{20}$ = __________%
8. 3 = __________%
9. $\frac{7}{8}$ = __________%
10. $\frac{3}{10}$ = __________%
11. $\frac{23}{25}$ = __________%
12. $2\frac{1}{2}$ = __________%
13. $\frac{1}{4}$ = __________%
14. $\frac{5}{8}$ = __________%
15. $\frac{1}{25}$ = __________%
16. $\frac{1}{3}$ = __________%
17. $5\frac{1}{4}$ = __________%
18. $\frac{19}{25}$ = __________%
19. $1\frac{1}{2}$ = __________%
20. $\frac{3}{8}$ = __________%

which answers are incorrect. Figure 20.9 shows another variation of this type of puzzle for younger children. The puzzle contains pairs that childen connect—a numeral and a set picturing that written numeral.

Word Puzzles

Word puzzles have two sections: the computational section and the puzzle-solving section. Figure 20.10 shows an example. The student works a series of problems identified on the worksheet with letters of the alphabet. When he obtains an answer, he searches for it in the puzzle-solving section. If he finds it, he places the letter associated with it in the blank. When the puzzle is correctly completed, phrases or names are spelled out.

FIGURE 20.9
A dot-numeral puzzle.



Name ________________

Connect the dots and the numbers that go together with lines.

Cross-Number Puzzles

Cross-number puzzles are like crossword puzzles except that the clues are mathematical problems and the answers are numbers that fit into a grid. The problems may take almost any form: verbal problems, division of multidigit numbers, converting expanded notation to numerals, and so on. Figure 20.11 shows an example of a puzzle designed to practice numeration skills.

COMPUTERIZED DRILL

Literally hundreds of available commercial programs provide computerized mathematics drill. This type of computer-assisted instruction is the easiest to write; it requires no knowledge of teaching methods or curriculum and can be done with computers having small memories and no disk drives. Thus, many companies offered such programs as their first "instructional" software in the prehistoric days (around 1977) when microcomputers had small memories, before educators had developed guidelines for evaluating software, and when computers were so much of a novelty that children were motivated to work harder doing drill on computers than they were using pencil and paper only. If you have the resources to use your computers for driling students (including the money for expensive software), consider some of the newer programs that incorporate drill into arcade-type games or simulations.

FIGURE 20.10
A word puzzle.

Name__________________

King Tongo, Chief of an African tribe, received a solid granite throne as a gift. He decided to store it in the attic of his thatched hut. But, the weight caused the house to collapse. Moral?

100 200 45 100 $66\frac{2}{3}$ 200   40 75 45

$66\frac{2}{3}$ $87\frac{1}{2}$ 30 200 $87\frac{1}{2}$ $62\frac{1}{2}$ 25 150 50 90 90

75 45 $77\frac{7}{9}$ 90 200 90   90 75 45 $77\frac{7}{9}$ $66\frac{2}{3}$ 5

$62\frac{1}{2}$ 45 125   90 125 45 40   125 75 150 45 $62\frac{1}{2}$ 200 90

Directions: Work the problems. Look for the answer above and write the letter of the problem in the blank above the number. When you are finished, the answer to the question will appear.

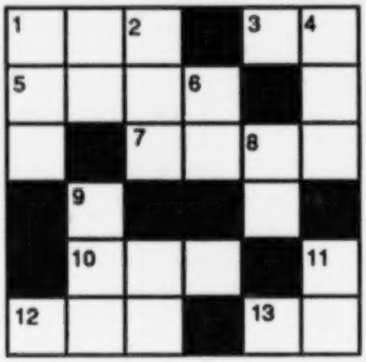
A. ____ % of 30 = 15
B. ____ % of 75 = 25
C. ____ % of 90 = 9
D. ____ % of 64 = 3.2
E. ____ % of 19 = 38
F. ____ % of 56 = 7
G. ____ % of 4 = 1
H. ____ % of 12 = 9
I. ____ % of 96 = 84
J. ____ % of 124 = 46.5
K. ____ % of 90 = 99
L. 10 = ____ % of 15
M. 12 = ____ % of 60
N. ____ % of 64 = 40
O. 36.0 = ____ % of 80
P. 42.5 = ____ % of 42.5
Q. 34.2 = ____ % of 36
R. 3 = ____ % of 2
S. What % of 10 is 9?
T. What % of 12 is 15?
U. What % of 99 is 77?
V. 22.5 is what % of 75?
W. 42 is what % of 105?
X. 125 is what % of 25?

PRACTICING MATHEMATICAL PRINCIPLES

Children not only need to practice computational procedures, they need to practice mathematical principles like the commutative property, the Pythagorean theorem, and the meanings associated with place value. For example, when you discuss the distributive property as it applies to the multiplication algorithm (Chapter 11), have students do exercises designed to clarify meaning and make them proficient with the property. Many textbooks include exercises like the following (the student is asked to fill in the blanks):

$$\begin{aligned} 5 \times 23 &= 5 \times (\underline{\quad\ } + 3) \\ &= (\underline{\quad\ } \times 20) + (\underline{\quad\ } \times 3) \\ &= 100 + \underline{\qquad} \\ &= \underline{\qquad} \end{aligned}$$

FIGURE 20.11
A cross-number puzzle



Objective: To provide symbolic practice with numeration concepts.

Materials: Pencil and paper

Across

1. 800 + 20 + 5 = ____
3. A number that has 3 tens, 7 ones ____
5. (1 x 10) + (3 x 100) + (7 x 1000) + 3 = ____
7. A number that has 5 thousands and 3 tens ___
10. The place value of the 9 in 987 is ___
12. The total value of the 8 in 179, 846 is ____
13. 130, 120, 110, 100, ____

Down

1. (8 x 100) + (7 x 10) + 0 = ____
2. 10 + 5 + 500 = ____
4. 674 rounded to the nearest hundred ____
6. 6237 = 6000 + 200 + ____ + 7
8. How many tens are in 350, including the hundred's place?
9. (1 x 10) + (8 x 100) = ____
11. The place value of the 6 in 3268 is ____

In geometry, after a lesson on area, students could practice using transparent grids to determine the areas of regular and irregular shapes.

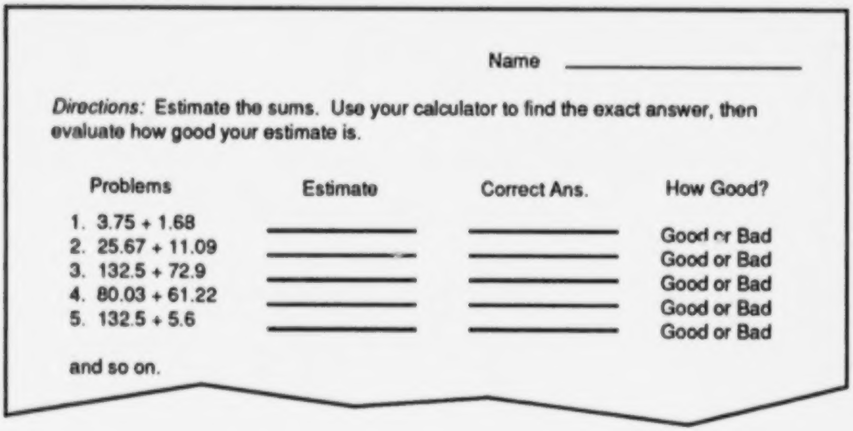
Applied Drill

Setting up drill situations using real-life applications of mathematics helps reinforce the relevance of the topics you are teaching. It also adds variety and spice to your classroom. For example, if you are studying percentages, have students calculate the tax on 10 items of their choice from a Sears catalog and fill out order forms. Subtraction problems could be practiced by making change in a store or restaurant. You might want to keep a notebook to record ideas for applying mathematics to a variety of real-life experiences. Group the ideas in your notebook according to the mathematical topics you cover in your class.

Estimation Drill

Estimation and mental arithmetic should be practiced like any other set of skills. The first step in developing the ability to estimate responses is understanding mathematical principles (what is multiplication; what do numerals like 235 mean). The next step is to help children develop estimation strategies, and finally to have them practice these strategies. Give students worksheets with problems to estimate and have them write their estimate. Accept their answer as correct if it is within certain parameters (perhaps within 25% of the correct answer). Start with problems that are easy to estimate and move to more difficult problems as the students become more proficient. Another strategy is to have students write their estimates, use their calculators to calculate the exact answer, then rate their estimate—was it a good or a bad estimate? Figure 20.12 shows the first few items of a worksheet that provides that type of drill.

FIGURE 20.12
An estimation drill.



Name ____________________

Directions: Estimate the sums. Use your calculator to find the exact answer, then evaluate how good your estimate is.

Problems	Estimate	Correct Ans.	How Good?
1. 3.75 + 1.68	________	________	Good or Bad
2. 25.67 + 11.09	________	________	Good or Bad
3. 132.5 + 72.9	________	________	Good or Bad
4. 80.03 + 61.22	________	________	Good or Bad
5. 132.5 + 5.6	________	________	Good or Bad

and so on.

RESOURCES FOR MATHEMATICS PRACTICE MATERIALS

Activity Resources Company, P.O. Box 4875, Hayward, CA 94540. (Math games, resource books, puzzles)

Creative Publications, P.O. Box 10328, Palo Alto, CA 94303. (Math puzzles, games, drill worksheets, computer games)

Creative Teaching Associates, P.O. Box 7766, Fresno, CA. 93747. (Math board games, card games, puzzles, worksheets)

Dale Seymour Publications, P.O. Box 10888, Palo Alto, CA 94303. (Puzzles, worksheets, applications practice, software)

Educat, Dept. EL, P.O. Box 2891, Clinton, Iowa 52735. (Their catalog contains advertisements from a variety of producers of practice materials)

Midwest Publications, P.O. Box 448, Pacific Grove, CA 93950. (Books and worksheets designed to develop thinking skills)

Most major journals or magazines for teachers include occasional or frequent articles about practice.

QUESTIONS AND ACTIVITIES

1. Name a television game show and describe how it could be adapted to do mathematical drill.
2. Name a popular board game (other than those discussed in this chapter) and describe how it could be adapted for mathematical drill.
3. Discuss classroom problems that could result from frequent use of timed drills.
4. Describe how some aspects of childrens' favorite television shows (e.g., "The Smurfs," "Dungeons and Dragons," or "Dukes of Hazzard") could be incorporated into a game.
5. Write a Logo program that drills students on addition facts.
6. Find a commercial mathematics game in the instructional materials section of your college library, in the school where you work, or in a teacher's supply store. Critique it using the Game Evaluation form (Figure 20.1) included in this chapter.

APPENDIX

A An Introduction to BASIC

BASIC has been called "the language of microcomputers." Some microcomputers come with BASIC built in so that it is available as soon as you turn on the computer. Others provide BASIC on diskette, requiring loading before you can start programming. A few computers like the IBM PC do not come "bundled" with BASIC—you must pay extra for a diskette containing BASIC. For all these machines, however, BASIC is the primary language.

BASIC was chosen as the language of microcomputers first of all because it is a relatively simple language with few commands. The program that translates BASIC into code that the computer can understand, therefore, does not take a great deal of computer space. Originally, microcomputers had a very small amount of memory, so neither the translating programs for more complex programming languages like Pascal or PL/1 nor programs written in these languages would fit into memory. In addition, BASIC's simple syntax means that a beginning programmer can start writing BASIC programs almost immediately without having to know how to do such things as use an operating system, compile a program, or declare variables—things that are required with more "sophisticated" languages. In BASIC you can write a one-line program to print something on the screen or to do a mathematical computation, whereas in Pascal or COBOL, a comparable program would consist of several lines and require that the programmer understand numerous concepts.

A second reason why BASIC was chosen as the language of microcomputers was that it is *interpreted.* With interpreted languages, as in the BASIC that comes with Apple II and IBM-type computers, programs are translated into machine language and executed one line at a time when they are run. With compiled languages, on the other hand, programs are translated into machine language all at once and stored in memory or on disk for later use. Once a program has been compiled, it does not need to be compiled again unless you wish to make changes in your program. Thus, compiled programs run much faster than interpreted programs. The disadvantage is that you have two versions of your program—a Pascal version (or whatever compiled language you are working with) and a machine-language version. Since early microcomputers did not have disk drives and since memory was small, there was no place to store the compiled program. Note that several newer, more powerful versions of BASIC, such as True BASIC and Turbo BASIC, are compiled rather than interpreted.

Dozens of different versions of BASIC are available—Atari BASIC, Applesoft BASIC, Microsoft BASIC, C BASIC, to name a few. Some of the dialects of BASIC are not much richer than when personal computers first were introduced in 1976. Other versions such as True BASIC and Turbo BASIC contain many features of more sophisticated languages, like PL/1 or Pascal. Although the fundamental 25 or so statements (words that tell the computer what to do) are virtually the same in all BASICs, some dialects of BASIC have between 25 and 75 additional statements to make use of the more powerful capabilities of the modern microcomputer. Some examples are graphics statements, statements to use disk drives, printer statements, and statements to interpret and manipulate words entered from the keyboard.

The following is a BASIC tutorial. It presents the commands that you need in order to do the activities in this textbook and to use the language as a tool in your classroom for helping children understand mathematics. You may be asked by your teacher to complete the enclosed activities independently or in small groups. It is not sufficient to read the material in this chapter, however. You should experiment on a computer with the ideas presented: enter and run the sample programs and try the exercises at the end of each section.

ENTERING AND RUNNING A BASIC PROGRAM

A program is a list of statements to be executed by a computer. We may ask the computer to do things such as printing something on the screen, finding the answer to a computation like Y * 2.3 and printing the results, or repeating something 25 times. Each of these tasks requires some sort of statement explicitly telling the computer to do that task. For example, the following statement prints the word "hello" on the screen.

```
10 PRINT "HELLO"
```

In BASIC, each line in a program has a line number. Typically, BASIC programmers number their lines by 10s so that if they must come back later and insert lines, there will be space for them. The following program illustrates the use of line numbers. It prints the result of a mathematical computation.

```
10 PRINT "THE ANSWER TO THREE TIMES FOUR IS"
20 PRINT 3*4
30 END
```

The first step when running a BASIC program is to turn on the machine and get BASIC up and running. On the Apple II and Commodore computers, BASIC is ready to run as soon as the machine is on. However, if you wish to save a program to or load a program from your disk drive, you must boot the system with the disk operating system (DOS) diskette. With IBM, Tandy, or similar computers, you may have to load BASIC from disk before you can run a program. Consult your computer manuals for directions on how to load BASIC if it is not there when the computer is turned on.

Next, type "NEW" (and hit the RETURN or ENTER button) to clear out any old program that might still be in memory. Now, type in the above three-line program, including the line numbers. If you make a mistake, hit the left arrow, delete, or rubout key (depending on your computer) to erase the mistake and retype the characters correctly. When you have finished the line, type the RETURN button and start the next line.

If you discover after you have completed a line and hit RETURN (or ENTER, depending on your computer) that it has a mistake, simply retype the line, line number and all. The computer will delete the old line and insert the new one in its place. If you discover that you have an extra, unwanted line in your program, you may delete the unwanted line by typing its number and then RETURN.

To see what your program looks like at any time, type LIST (no line number; this is an immediate command). If your program is longer than the number of lines that fit on your screen, use the following for listing parts of your program.

`LIST 10-1000`	*Lists lines 10 to 1000, inclusive*
`LIST -200`	*Lists up to line 200*
`LIST 100-`	*Lists from line 100 to the end of the program*
`LIST 40`	*Lists line 40*

Once your program is entered and has no errors (as far as you can tell), you may execute it by typing RUN (again, no line number). RUN, like any other immediate statement (a statement not inside a program, one that is executed as soon as it is typed and RETURN is hit), is not preceded by a line number.

PRINTING TEXT ON THE SCREEN

To print a string (a sequence of alphabetic, numeric, and special characters like punctuation marks), use the statement PRINT followed by a pair of quotation marks enclosing the desired output.

```
10 PRINT "MATH IS LOTS OF FUN"
20 PRINT "AND SO IS SCIENCE"
30 END
```

The print statement may also be used to print numerical values:

```
10 PRINT 35
20 PRINT 98
30 END
```

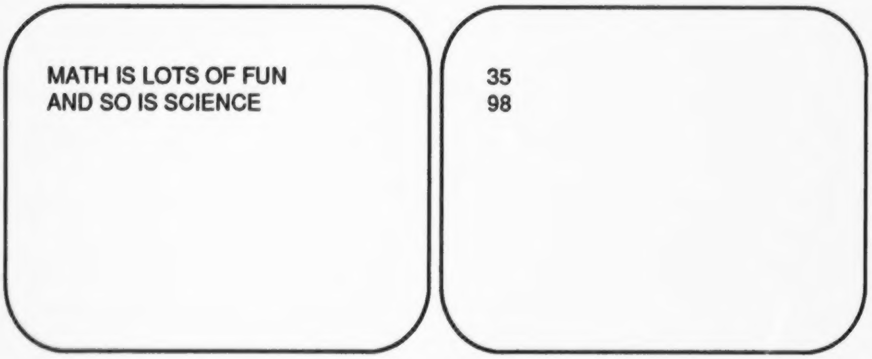
The statements in the above program are different from the following:

```
10 PRINT "35"
20 PRINT "98"
30 END
```

In this program, 35 and 98 are strings, rather than numbers. When running these two simple programs, there appears to be no difference in the output. Indeed, there is not. The difference is internal, in the way the computer stores these two pairs of values. In the first, the numbers 35 and 98 are stored as numerical values (in base two). You can perform arithmetic operations on numerical values but not on strings. The string values "98" and "35", therefore, may be printed but not subtracted, divided, and so on.

Each time a print statement is encountered, the string in quotes is printed, starting at the left-hand side of the screen (column 1). Figure A.1 shows what the two programs above would look like if typed in (type NEW before you type in any new program) and run.

FIGURE A.1



```
MATH IS LOTS OF FUN
AND SO IS SCIENCE
```

```
35
98
```

Often, we wish to have our output printed on a clean screen. To clear the screen and start the output at the top, left-hand corner, use one of the following, depending on the brand of computer you are using:

```
5 HOME  (Apple)
5 CLS   (IBM & Radio Shack)
5 PRINT "♥"  (Commodore 64 and 128)
```

Print statements are sometimes used to print graphics. Using the characters that appear on the keyboard, it is possible to generate some interesting pictures. The following program prints a crude "automobile."

```
5 HOME
10 PRINT "    /------ "
20 PRINT "   /        "
30 PRINT "XXXXXXXXXXXXXXXX)"
40 PRINT "XXXXXXXXXXXXXXXX)"
50 PRINT "   0        0"
```

Questions

1. Write a program that uses the print statement to print a paragra[illegible] about using bean sticks to teach whole number numeration. Clear the screen so that only your paragraph appears. Arrange your printout so that no words "wrap around" (that is, are split in the middle and continued on th[illegible] line).

2. Write a program that prints your first and last initial in large block letters, at least 2 inches high, on the screen.

The print statement will do much more than print simple strings. It will also do arithmetic and print the results. When the following program is run, for example, the computer prints the answer, 15, on the screen.

```
5 HOME
10 PRINT 20 - 5
20 END
```

All the basic arithmetic operations can be done on the computer. The symbols used to tell the computer to carry out arithmetic operations are as follows:

```
+ Add
- Subtract
* Multiply
/ Divide
^ Exponent
```

The order in which arithmetic operations are carried out by BASIC are the same as in algebra: exponents are done first, multiplication and division are done next from left to right, and finally, addition and subtraction are done from left to right. Examples of how operations are carried out are shown in Figure A.2. Notice that parentheses override the normal hierarchy of operations.

FIGURE A.2

$$10\ \text{PRINT}\ 3 + \underbrace{4 * 5}_{20} - \underbrace{8 / 4}_{2}$$

$$\underbrace{\underbrace{3 + 20}_{23} - 2}_{21}$$

$$20\ \text{PRINT}\ (5 * \underbrace{(3 + 2)}_{5 * 5}\,)\ /\ (2 + 3)\ \hat{}\ 4$$

$$\underbrace{5 * 5}$$

$$25 / \underbrace{(2 + 3)}\ \hat{}\ 4$$

$$25 / \underbrace{5\ \hat{}\ 4}$$

$$25 / 625$$

$$.008$$

Questions

1. Suppose you borrowed $2400 at 12% interest to get you through your senior year of college. Write a program that will calculate and print out the amount you will have to return to the bank when paying off your debt a year later.

2. Write a program to print out the average score for each of three students who scored as follows on four exams (worth 100 points each): Student 1—89, 90, 82, 93; Student 2—75, 72, 88, 85; Student 3—99, 90, 82, 94.

Usually, we want to make our printouts look attractive. To do this, there are several formatting techniques available in BASIC.

We may put several things on a print line if we separate them with semicolons.

```
10 PRINT 27; " TIMES "; 35;" EQUALS "; 27*35
```

will print

```
27 TIMES 35 EQUALS 945
```

Notice the spaces inside the quotes. Without them, the output would look like this on most computers (some automatically leave spaces in front of and/or behind numbers):

```
27TIMES35EQUALS945.
```

Spaces outside of quotes are ignored in BASIC.

Some computers will allow you to get away without the semicolons to separate multiple statements on a print line, but it is a good idea to put them in since there are certain situations where you can get in trouble if you ignore them.

Commas are sometimes used in print lines to place strings or numbers in columns. Each time a comma appears in a print line, the next value to be printed is printed in the next screen print field. There are between 3 and 5 print fields on the screen, depending on the type of computer you have. Observe the output of the following program:

```
10 PRINT "AVERAGES FOR THIRD PERIOD CLASS"
20 PRINT "NAME","TEST 1","TEST 2","AVERAGE"
30 PRINT "JOHN",35, 50,(35+50)/2
40 PRINT "JUANITA",44,49,(44+49)/2
50 PRINT "JIM",26,45,(26+45)/2
60 END
```

Another way to format output is to use TAB statements. The TAB statement works the way a tab does on a typewriter—it causes the printing to start on a specific column on the screen. The statement

```
10 PRINT TAB(10) "HELLO"
```

prints HELLO starting in column 10.

```
10 PRINT "JIM";TAB(15)26;TAB(30)45;TAB(40)(26+45)/2
```

prints JIM at column 1, 26 at column 15, 45 at column 30 and 35.5 at column 40 (on 40 column machines, it prints 35.5 at the beginning of the next line). The advantage of TABs over the use of commas is that you, rather than the computer, can control where your columns go. Also, computers like the Apple have only 3 print fields and if you wish to print in 4 or more columns on the screen, or if you wish to design a specific format for your output (e.g., if you are designing CAI for your students), the comma will be inadequate. The following program is similar to the program above except that it uses TABs to format the output and adds more scores.

```
10 PRINT "NAME    TEST 1    TEST 2    TEST 3    AVERAGE"
20 PRINT "JOHN";TAB(9)35;TAB(18)50;TAB(27)44;TAB(36)
   (35+50+44)/3
30 PRINT "JUANITA";TAB(9)44;TAB(18)49;TAB(27)47;
   TAB(36)(44+49+47)/3
40 PRINT "JIM";TAB(9)26;TAB(18)45;TAB(27)39; TAB(36)
   (26+45+39)/3
50 END
```

USING VARIABLES

Everything that we have done so far with BASIC could be easily duplicated with a typewriter and/or a calculator. One of the most important features of a computer that sets it apart from the calculator is the ability to assign a value to a memory location and to use that value later in computations. In addition, the program can modify that memory location as a program is being executed. This powerful feature allows us to do some complex, multistep arithmetic very quickly and easily on the computer.

In most versions of BASIC, a memory location (often called a *variable*) is named by one letter (X), two letters (ME), or a letter followed by a number (X1). Other versions of BASIC allow up to 40 characters in a variable name. One way to put a value in a memory location is to use an *assignment*. An *assignment statement* has a variable or memory location name on the left side, an equal sign and a single value or a sequence of values or variables on the right side. Some BASICs require a LET statement in front of the assignment, but in most, the LET statement is optional.

```
10 LET A=5
20 B1=5*6
30 CC=B1 + 22.3 - (8 + 3.2)
40 PRINT A,B1,CC
50 END
```

In this program, line 10 assigns 5 to memory location called A, line 20 multiplies 5 by 6 and assigns the result to B1, and line 30 assigns 41.1 to CC. Line 40 prints the contents of each of these memory locations in a print field. Note that nothing is printed on the screen when an assignment is made. If line 40 were eliminated from the program, nothing would appear on the screen at all when the program is run. The actions on lines 10 to 30 occur in the computer's memory only. On the other hand, line 40 does nothing to the computer's memory; its purpose is to change the appearance of the screen.

String values may also be assigned to memory locations. When assigning strings, use a $ sign after the variable name and place the string in quotes.

```
10 A$=" JOHN "
20 B$="IS"
30 CC$=" FAST."
40 PRINT A$;B$;C$
50 END
```

Questions

1. Write a BASIC program that calculates the circumference of three different circles. Assign the radius of each circle to the variables R1, R2, and R3, then use these variables to calculate and print out the circumference of each in the following form:

```
CIRCUMFERENCE 1 =
CIRCUMFERENCE 2 =
CIRCUMFERENCE 3 =
```

2. Write a BASIC program that calculates and prints the amount of money that Sally, June, and Joe need to pay in a restaurant for meals costing $12.75, $8.44, and $10.27, respectively. Include a tip of 15% (based on the food only) and 5 1/2% tax (also based on the food only) for each meal.

UNCONDITIONAL AND CONDITIONAL BRANCHING

When a program is executed in BASIC, it is evaluated and implemented line by line, from the smallest line number to the largest unless there is a statement telling the computer to change the sequence. One such statement is GOTO. GOTO is an unconditional branch, telling the computer to jump to a given line number. The following program makes use of GOTO to print the sum of all numbers X and smaller.

```
10 X=0
20 S=0
30 X=X+1
40 S=S+X
50 PRINT X,S
60 GOTO 20
70 END
```

In this program, lines 10 and 20 *initialize* the variables X and S. Line 30 adds 1 to the contents of X, and line 40 adds the value in X (1,2,3,4,5, . . .) to S which holds the sum of values up to X.

Line 60 in this program is an unconditional branch. When you execute this program, it will keep running until you break the program by hitting the break button (Control-C on Apple computers, CTRL-BREAK on IBM-type computers, and RUN-STOP on Commodore computers). Figure A.3 is a flow chart of the program.

FIGURE A.3

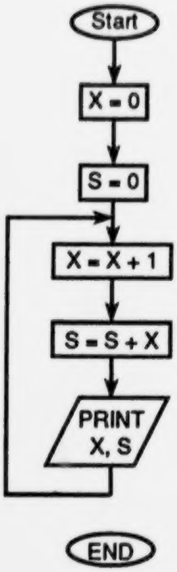
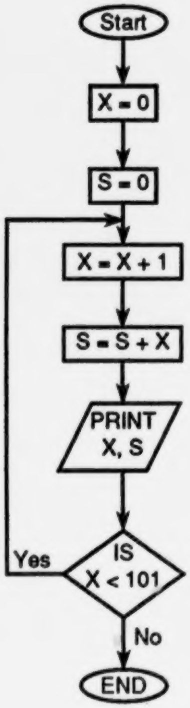


FIGURE A.4



Notice that the program contains a *loop,* a sequence of statements that are repeated over and over. In this program, no way is provided out of the loop. To stop execution of the loop, you must intervene from outside by turning off the computer or hitting "break" (CTRL-C, etc.).

Fortunately, there is a command that will allow us to check the value of a memory location, then branch or not branch, based on its contents. That statement is IF . . .THEN. If, for example, we wanted to only find the sum of all numbers up to 100, we could change line 60 in the program above to a conditional rather than an unconditional branch.

```
60 If X<101 THEN30
```

This statement compares the contents of X to 101. If X is less than 101, the program goes to line 30. Otherwise it goes to line 70 and stops.

Figure A.4 shows the flowchart of this program with the conditional branch.

The general form of the IF . . . THEN statement is

`IF X = 4 * Y THEN`
`IF` *boolean expression* `THEN` *expression*

The Boolean expression is a statement that evaluates a line as true or false. Some of the possibilities follow:

`IF X<5`	(X less than 5)
`IF MM>=25`	(MM greater than or equal to)
`IF Y<=13`	(Y less than or equal to 13)
`IF Z2<>17.3`	(Z2 not equal to 17.3)

After the THEN statement, you may place any valid BASIC statement(s). For example, you may place a GOTO (or implied GOTO that consists of a line number without a GOTO), an assignment, or even another conditional branch.

```
IF X>3 THEN PRINT "HELLO"
IF Z2<>17.3 THEN 45 (or THEN GOTO 45)
IF Y<=13 THEN IF Z=3 THEN 1000
```

If the Boolean expression is true, the THEN part of the statement is executed, all the way to the end of the line. You may wish to include several statements after the THEN; each statement must be separated from the others by colons and be a self-standing, complete statement (one that normally would have a line number and be on a line by itself).

```
IF Z=6 THEN PRINT "CORRECT":A=7:GOTO 100
```

The following is a similar program that calculates the sum of the numbers from 1 to 100, then prints the result (only once).

```
10 S=0
20 X=0
30 S=S+X
40 IF X<100 THEN X=X+1:GOTO 30
50 PRINT "THE SUM IS ";S
60 END
```

Questions

1. Write a program that will find the sum of all even numbers between 100 and 200.
2. Write a program that will print each number between 1 to 100 and its square on the same line.

INPUT FROM THE KEYBOARD

Another powerful feature of BASIC is the ability to write a program that will accept input from the keyboard. For example, you could write a program that would allow the user to type the lengths of the legs of any right triangle and have the computer calculate and print out the length of the hypotenuse. The statement used for this is the INPUT statement. When the computer encounters an INPUT statement, it stops, prints a question mark on the screen and waits for an input from the keyboard. The user types a number or a string and hits return. The computer then continues executing the program. The following line is the simplest form of the INPUT statement:

```
10 INPUT A
```

Whatever number the user types is stored in A for later use. If the user typed 5 and hit return, the result would be exactly the same as the assignment statement, A = 5.

The following program uses INPUTs to get the length of the base and altitude of triangles, then calculates and prints the area of each triangle. Note that in this program A$ and A are two different variables—A$ is a string variable and A is a numeric variable, and the contents of each are stored in a different part of memory.

```
10 PRINT "PLEASE TYPE YOUR NAME"
20 INPUT A$
30 PRINT "HELLO ";A$;". IF YOU TYPE THE LENGTH"
40 PRINT "OF THE BASE AND ALTITUDE OF TRIANGLES"
50 PRINT "I WILL CALCULATE THEIR AREAS."
60 PRINT "TO STOP, TYPE 99 FOR THE BASE."
70 PRINT "NOW TYPE THE BASE OF A TRIANGLE"
80 INPUT A
90 IF A=99 THEN 140
100 PRINT "TYPE THE ALTITUDE OF THE TRIANGLE"
110 INPUT B
120 PRINT "THE AREA  IS ";1/2*A*B
130 GOTO 70
140 END
```

Questions:

1. Write a program that will find the volume of any rectangular solid after the height, length, and width are entered.
2. Write a program that will allow you to input the amount of a meal in a restaurant, the percent tip (as a decimal), and the tax. The computer then calculates and prints the amount of your bill.

ANOTHER METHOD FOR LOOPING

A second, and more efficient method for looping is the FOR TO, NEXT statement. This statement defines the beginning and end of a loop and has a built-in counter. The following program prints HELLO 10 times.

```
10 FOR X=1 TO 10
20 PRINT "HELLO"
30 NEXT X
```

Line 10 starts counting at 1 and ends at 10. Line 30 defines the end of the loop. Everything between lines 10 and 30, thus, is repeated 10 times.

With FOR TO, NEXT it is possible to count by 2s, 5s, or any other number. It is even possible to count backwards. To do this, add STEP and the increment value to the FOR line. The following program prints out the even numbers between 100 and 200.

```
10 FOR M=100 TO 200 STEP 2
20 PRINT M
30 NEXT M
```

This next program counts backward from 50 to 0 by halves, and prints the square root of each number counted. Note that when counting backwards, the first value in the FOR statement must be larger than the second value.

```
10 FOR ZZ=50 to 0 STEP -.5
20 P=SQR(ZZ)
10 PRINT ZZ, P
30 NEXT ZZ
```

Notice the built-in function, SQR. It allows you to determine the square root of any (positive) value. PRINT SQR(25) will print out the square root of 25.

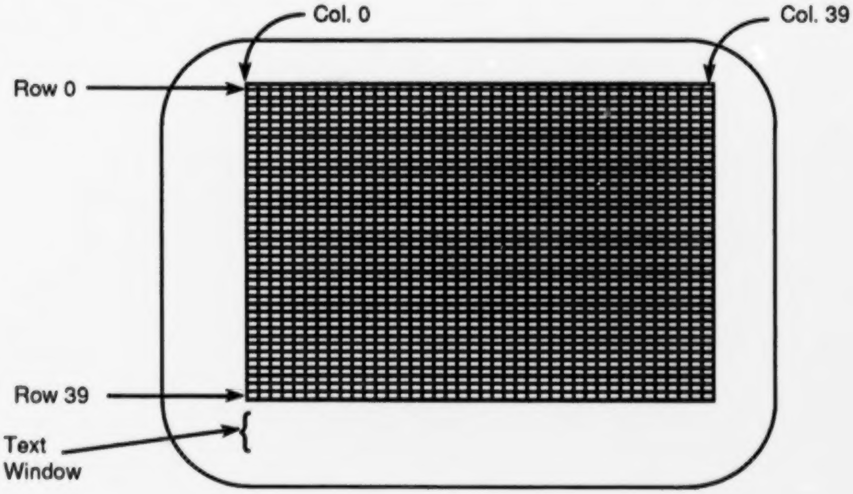
Questions

1. Write a BASIC program using FOR TO, NEXT that finds the sum of all the numbers that are divisible by 3 between 50 and 100.
2. Write a program using FOR TO, NEXT that will allow the user to input the radius of any circle, then will calculate and print the circumference and area of that circle.

APPLESOFT BASIC GRAPHICS

The Apple II family has two graphics systems (low resolution graphics and high resolution graphics), each using a different set of commands. To activate low resolution graphics, type the command GR either in the immediate mode or in a program. (The TEXT command will return the screen to the text mode.) The top ⅚ of the screen is now dedicated to graphics while a text window of four lines is at the bottom of the screen. The graphics portion of the screen consists of a 40 by 40 "grid," each square of which may be illuminated in one of 16 colors using the PLOT command. (See Figure A.5.)

FIGURE A.5

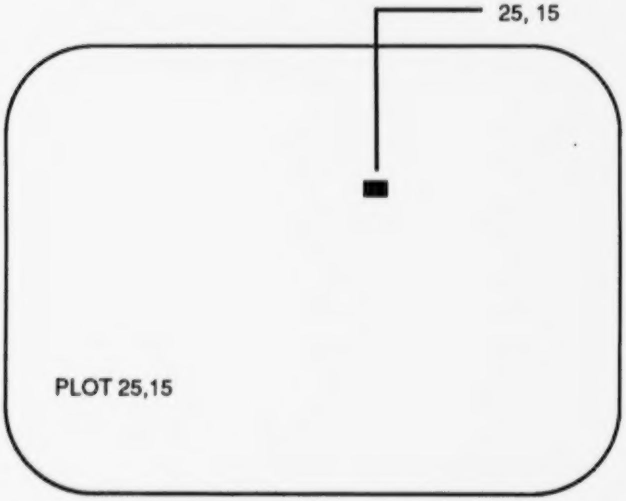


When you use the PLOT command, you need to follow it with two numbers; the first indicates the column (0 – 39) and the second the row (0 – 39) of the square you are turning on. The line,

```
30 PLOT 25,15
```

will plot the square in column 25, row 15 (see Figure A.6).

FIGURE A.6



In low resolution graphics, there are 16 colors available. To set a color, use the COLOR command. The following BASIC line sets the computer to color 5 (grey).

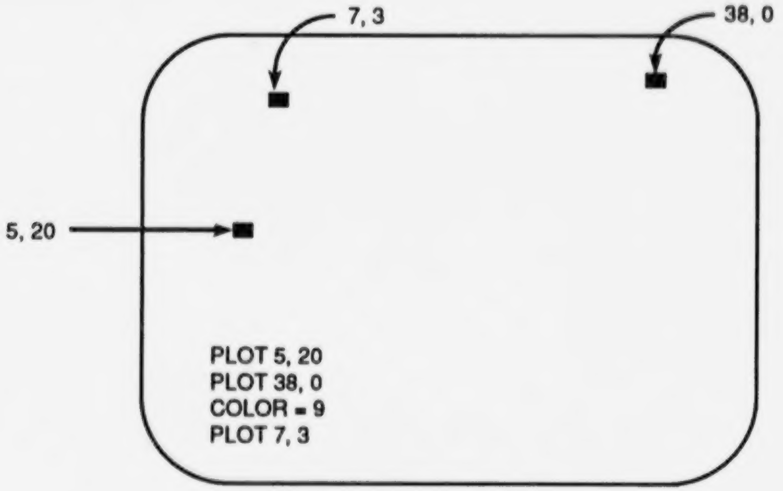
```
COLOR=5
```

Note that if you PLOT a point without first setting a color, the point will be plotted in the background color (black) and will not be visible on the screen. After you have used the COLOR command, all subsequent graphics commands will be drawn in the color specified until you change to another color.

The following program illustrated in Figure A.7 plots 2 squares in color 3 (purple) and one in color 9 (orange).

```
10 GR
20 COLOR = 3
30 PLOT 5,20
40 PLOT 38,0
50 COLOR = 9
60 PLOT 7,3
70 END
```

FIGURE A.7



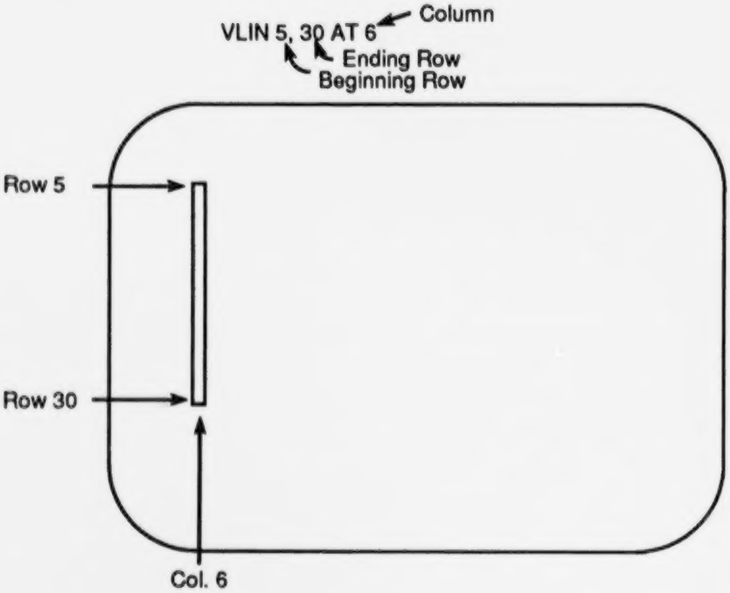
The following is a list of the colors and accompanying codes in low resolution graphics.

0	Black	8	Brown
1	Magenta	9	Orange
2	Dark Blue	10	Grey
3	Purple	11	Pink
4	Dark Green	12	Green
5	Grey	13	Yellow
6	Medium Blue	14	Aqua
7	Light Blue	15	White

There are two other low resolution graphics commands: the VLIN command which draws vertical lines (see Figure A.8) and the HLIN command (see Figure A.9) which draws horizontal lines. The format of the VLIN command is as follows:

```
VLIN 5, 30 AT 6
```

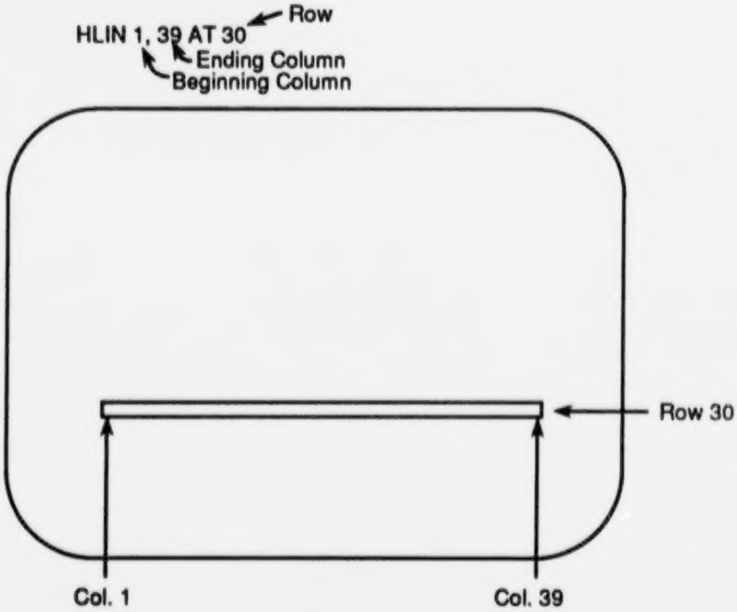
FIGURE A.8


VLIN 5, 30 AT 6
Column
Ending Row
Beginning Row
Row 5
Row 30
Col. 6

The format of the HLIN command is similar:

```
HLIN 1,39 AT 30
```

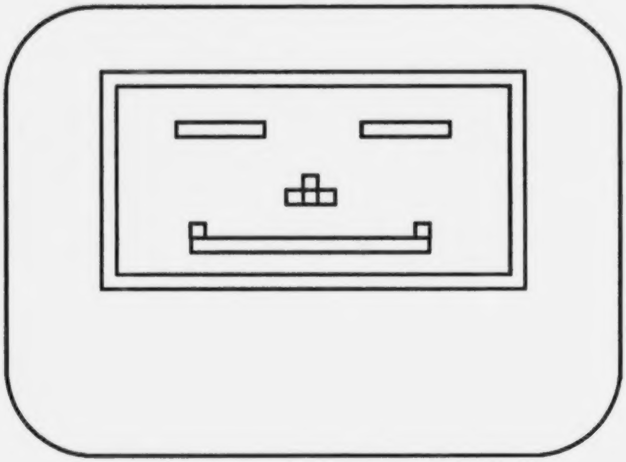
FIGURE A.9


HLIN 1, 39 AT 30
Row
Ending Column
Beginning Column
Row 30
Col. 1
Col. 39

The following program puts a smiling face on the screen, as in Figure A.10.

```
10 GR
20 COLOR = 14
30 HLIN 10,30 AT 10
40 HLIN 10,30 AT 30
50 VLIN 10,30 AT 10
60 VLIN 10,30 AT 30
70 HLIN 15,18 AT 14
80 HLIN 22,25 AT 14
90 PLOT 19,21:PLOT 20,20:PLOT 21,21
100 HLIN 15,25 AT 25
110 PLOT 15,24:PLOT 25,24
120 END
```

FIGURE A.10



The second type of Applesoft graphics is the high resolution system. This system has only three commands, the HGR command which places the computer in the high resolution graphics mode, the HCOLOR command which sets colors, and the HPLOT command which plots points or lines.

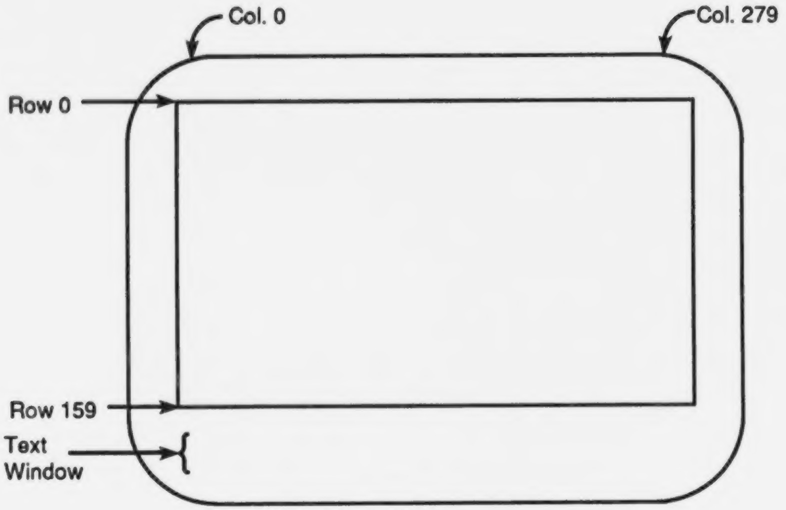
The colors in high resolution graphics are as follows:

0	Black	4	Black
1	Green	5	Orange
2	Violet	6	Blue
3	White	7	White

The high resolution screen has 280 (0 - 279) columns and 160 (0 - 159) rows with a four-line text window at the bottom. (Note: if you use the HGR2 command rather than HGR, the entire screen is used for graphics—a 280 by 192 grid with no text window at the bottom.) To plot a point in the middle of the screen as in Figure A.11, use the command

```
HPLOT 140,80
```

FIGURE A.11



This will be a point 140 columns over and 80 rows down. As with low resolution graphics, you must set a color other than black to see it on the screen. To draw a line, specify the endpoints in the HPLOT command. The following draws a line from the point 0,0 to the point 100,150. Note that in high resolution graphics you may draw diagonal lines, something you cannot do in low resolution graphics.

```
HPLOT 0,0 TO 100,150
```

High resolution graphics allow you to string as many lines together as you want with one HPLOT, so long as there are no more than 255 characters between the beginning of the line and the RETURN character (the end of the line). The following program, for example, draws a triangle:

```
10 HGR
20 HCOLOR=6
30 HPLOT 50,100 TO 150,50 TO 200,100 TO 50,100
40 END
```

Questions

1. Draw a scene in low resolution graphics.
2. Draw a square, rectangle, parallelogram, and trapezoid in high resolution graphics.

OTHER GRAPHICS SYSTEMS

Versions of BASIC for other microcomputers all have their own graphics commands, all somewhat different from others. The BASIC available for IBM-type computers has three levels of resolution and has commands for drawing lines, circles, squares, and other shapes. Consult your reference manual or tutorial for information about the graphics system available on your microcomputer.

SUGGESTED READINGS

Lien, D. A. *BASIC Programming for the IBM Personal Computer.* Dubuque, IA: Wm. C. Brown, 1984.

Culp, G. and Nickles, H. *An Apple for the Teacher.* Monterey, CA: Brooks/Cole, 1983.

Thompson, J. *Practical BASIC for Teachers.* Columbus, OH: Merril, 1985.

APPENDIX

B AppleWorks Spreadsheet

A spreadsheet program is used primarily for analyzing numerical data such as student grades or financial records. In a spreadsheet, data, labels, or formulas are entered into the cells (grid locations); the labels identify rows and columns of data and formulas are used to operate on the data. This appendix looks specifically at the AppleWorks spreadsheet but there are numerous other spreadsheets for Apple and the other personal computers, such as Lotus 1-2-3, SuperCalc, or MultiPlan.

STARTING A NEW SPREADSHEET

1. Boot AppleWorks.
2. Select Option 1 on the Main Menu (**Add files to desktop**), then move the highlight to Option 5 (**make a new file for the: Spreadsheet**). Hit <R> again to select **From scratch**.
3. Type the name of the new file (e.g., **Grades**)
4. The spreadsheet will now appear on the screen and cell A1 will be highlighted. Whatever you type at this point will be entered into that cell.

ENTERING LABELS, DATA, AND FORMULAS

A **label** is like a "string" in BASIC or Logo. It is a sequence of alphabetic or numeric characters. You may use a label as a column or row heading, or you may want to place the name of a person, her social security number, her address, and so on, into different cells. Here again you would use a label.

1. To enter a label, simply start typing alphabetic characters. If your first character is a number, it will be considered data; therefore, if you wish to use a la-

bel that starts with a number (e.g., 1/30/86), you must first type a quote ("). The quote also allows you to insert blanks before the beginning of the label (without a quote, a beginning space is not allowed). Note that labels are left-justified within cells.

2. To enter numeric data, simply type the number you wish to enter and either hit <R> or move the highlight to the next cell. Note, numbers are right-justified.
3. When a formula is stored in a location in the spreadsheet, only the value it calculates is displayed in the spreadsheet (unless you type OA-Z). The formula itself is displayed below the spreadsheet. A formula consists of numbers, arithmetic operations (+ − * /), cell names (A6 or B25), and/or functions. They must begin with a plus sign (+), a minus sign (−), a decimal, the digits 0–9, the left parenthesis, or the at sign (@). They may not begin with a letter. Some examples of formulas follow:

```
+B6*C4
(C8-C7+C6)/3
@SUM(C1..C5)/@COUNT(C1..C5)
```

Note: "@SUM" and "@COUNT" are functions that act on the row or column specified. This will be discussed later.

Activity

1. Enter the following spreadsheet.

```
Name            Test 1      Test 2      Test 3     Average
Smith, J.           88          90          76
Jones, B.           79          72          88
Ching, L.           92          95          89
Gomez, P.           95          82          85
Saul, T.            67          83          71
Ponsy, G.           82          37          67
Last, N.            90          92          84
   Avg.
```

2. Insert formulas for finding the average test score for each student and the average score on each test. Note, one way to do this is to specify each cell to be added (e.g., (B2 + C2 + D2)/3).

MOVING THE CURSOR IN THE SPREADSHEET

You may move around in the spreadsheet in several ways. The left, right, up, and down arrows move you one cell to the left, right, etc. OA→ moves you one screen to the right, OA← one to the left, OA↑ one screen up, and OA↓ one screen down. The Open Apple key together with a single digit (OA-1, OA-2 . . . OA-9) moves you proportionally through the spreadsheet—OA-5 puts you at about the middle and OA-9 puts you at the end.

If you wish to change an entry after it has been recorded by typing <R> or by moving the highlight, move the highlight to the desired location and retype the entry. Hit <R> or move the highlight to accept the entry (it will not be recorded until you do this). Note: if you wish to change the *type* of an entry in a cell (e.g., from a label to a number), you must move the highlight away and back again.

FUNCTIONS

These are codes that stand for common complex computations. For example, @SQRT will find the square root of a number. Functions are always preceded by an *at* sign (@) and followed by a cell name, list of names, or value. An example is @SUM (V13 . . . V25). This will calculate the sum of all values in column V, from row 13 to row 25. Some of the more common functions follow:

`@AVG(A1...A24, A27...A45)`	
`@COUNT(M5...M25)`	*counts non-blank entries*
`@INT(A2)`	*prints the integer portion of value*
`@MAX(M2...M30)`	*the largest value in the list*
`@MIN(M2...M30,N2...N30,P2...P30)`	*smallest value*
`@SQRT(25)`	*square root of value*
`@SUM(B2...Z2)`	*sum of the list*
`@IF(E1)90,1,2)`	*if first expression true, prints first value, otherwise, prints sec*

Activity

1. Replace the formula in the average column in Activity 1 (Column E) with a function (@AVG).
2. Blank out the rest of Column E and Row 9 (type OA-B, then select ENTRY, ROWS, or COLUMNS).

REPLICATING FORMULAS

One of the nicest features of a spreadsheet is that you can replicate a formula to other locations, using relative locations. For example, if you used @AVG (A1...E1) to calculate the average of the scores in Row 1, you may, with a few key strokes, repeat this same formula for as many rows as you wish. With the "relative" option, the formula placed in Row 2 will be @AVG (A2...E2), the formula in Row 3 will be @AVG (A3...E3), etc.

To replicate a formula, place the cursor on the cell containing the formula you wish to replicate. Press OA-C (for "copy"), hit <R> to select "Within Worksheet", hit <R> (the "source" should already be highlighted), move the highlight to the cell where you wish the replicated formula to begin and type a period. Now, move the highlight to the last cell where the formula will be and hit <R>.

Activity

1. Replicate the formula in E2 to the rest of the locations in Column E.
2. In Row 9, put formulas for printing the highest score on each test (using the replication feature).
3. In Row 10, put formulas for the lowest score on each test.

CHANGING CELL WIDTHS

AppleWorks allows you to change the width of cells so that you can put in longer names. To change the cell widths, type OA-L (layout). Select **Rows** or **Columns**, then highlight all the rows or columns you wish to change and hit <R>. Move the highlight to **Column Width**. Hold down the OA key and use the left or right arrow keys to change the column width.

CHANGING VALUE FORMAT

With the layout option, you may specify the number of decimals that will be printed in a cell, row, or column and whether it will be printed in percent format, dollar format, or so forth. Type OA-L, then select **Entry, Row, Column,** or **Block**. Hit <R> to select **Value Format**. If you choose **Fixed**, you may position a fixed decimal from 0 to 7 spaces from the right.

CHANGING LABEL FORMAT

You may also change the layout of labels. Type OA-L, move the highlight to Label Format, then select **Left Justify, Right Justify,** or **Center** (you may also protect a cell so that it will not inadvertently be changed).

RECALCULATION

Although, when you enter a value into a cell in AppleWorks, any formulas that use that value are automatically recalculated, you may sometimes want to manually ask AppleWorks to recalculate. An example is when you have sequences of cells each dependent on a previous value. Select OA-K, then choose **Calculation by Rows** or **Calculation by Columns**.

INSERTING OR DELETING ROWS OR COLUMNS

To delete a block of the spreadsheet, type OA-D. To insert rows or columns, type OA-I.

Activity

1. Increase the width of Column 1 (names) by five spaces.
2. Print Columns 2 through 4 in fixed format with two decimal places.
3. Center the labels on Columns 2 through 5.
4. Print Column 5 to three fixed decimal places.
5. Insert another name and test scores for that person in your spreadsheet.

PRINTING YOUR SPREADSHEET

You may view or change the printer options as you did with the word-processing program. Type OA-O and change the values by typing the two-letter code (usually, followed by a number). For example, type OA-O, CI, then 17 to print your spreadsheet seventeen characters per inch. Once the printer options are set up as you wish, type OA-P to print. You may at this point highlight part of the spreadsheet to print or print out the whole thing. Next, select the appropriate printer and the number of copies (of course, make sure that your printer is turned on and on line).

Activity

1. Print out your spreadsheet.
2. Experiment with the printer options (e.g., print it out at 17 characters per inch) and reprint the spreadsheet.

APPENDIX

C AppleWorks Data Base

A data base program is designed to allow you to enter, store, and manipulate information such as names, addresses, schedules, inventories, and comments. A file of information (created with the data base program) should contain a set of information that logically goes together. For example, you would set up one file to keep track of all students in your class and another to keep track of your collection of record albums.

Before you start entering data into a data base file, it is important to have carefully planned the format of your file. One reason is that the first thing you do when setting up a data base is to specify your categories of data. It is more difficult to add categories later, after you have begun to enter data. In addition, if you have already set up your categories and entered data for 20 records and you add another category, you will have 20 records with a blank category.

In addition to planning the format of your file, you need to decide the codes and the types of names and values you will enter into the categories. For example, if one of your categories is "Marital Status," you might decide that the four possible entries are "M," "D," "W," and "S" for married, divorced, widowed, or single. If you are not consistent in the codes or formats you use, you will find it very difficult to manipulate your data after it is entered (e.g., if you use several ways of indicating that a person is married, you have to remember and type each of those to select all people from your file who are married).

CREATING DATA-BASE FILES

Creating Categories and Entering Data

Boot AppleWorks, select **Add Files**, select **Make a new file for the Data Base** and **From scratch.** When you type the name of your new file, the category-create screen will appear. Type in your category names (limit, 20 characters). To add, delete or change categories, you may use the up and down arrows to move between categories. Type OA-I to insert a category and OA-D to delete a category. You may use the insert or overstrike mode (use OA-E to switch between the two) to make changes in the category names. Note: if you use category names containing "date" (e.g., Birthdate) or "time," Appleworks will convert entries in that category to the standard date and time formats.

When you are finished, hit <ESC>. The program will now go directly into the "Insert New Records" mode. At this point, you may type in the information for each category (you may type as much information as will fit on one line). When you have completed an entry, hit <R>. If you wish to leave a category blank, hit <R>. When you reach the end of a record, you will automatically be put at the beginning of the next one. If you do not wish to create a new record, hit <ESC>.

Multiple vs. Single-Record Layout

In the single-record layout, you look at all the information stored in one record. In the multiple-record layout, you can see some of the categories of up to 15 records at one time. To switch between the two modes, type OA-Z. The single-record format is used when entering data.

Inserting a Record

To insert a record into your file, position the cursor in the record that follows the record you would like to add. Hit OA-I and a blank record will appear. Insert as many records as you want. Press <ESC> to return to the place where you were before you pressed OA-I.

To add records to the end of the file, use the down arrow key to move the cursor past the last file. AppleWorks will ask if you want to insert new records. Choose **yes** or press **y**.

Activity

1. Create a new data base file with the following categories:

    Last Name
    First Name
    Street Address
    City
    State
    Zip Code
    Telephone Number
    Birthdate
    Sex
    Ethnicity
    Reading Group
    Math Group

2. Decide the types of entries you will use for the last four categories and enter data for 10 records. Since we will be selecting and sorting, etc., in later activities, your category names should be consistent.

Editing Entries

1. To delete one character to the left of the cursor press <DELETE>.
2. If you are inserting, deleting, or otherwise changing characters and decide that you want to cancel the changes you have made, hit <ESC>. To accept the changes, hit <R>.
3. CONTROL-Y followed by <R> will delete all characters from the cursor positon to the end of the category.

Dates and Times

If a category contains the word *date*, AppleWorks will convert almost any date you enter to the standard format. The standard format for a date is Sept 4 83. If you type 2/4/86, it will be changed to Feb 4 86. The same is true for 2.4.86, 4February1986, and several other formats. This allows AppleWorks to easily arrange records by dates.

Similarly, AppleWorks has a standard format for time and will convert any entry in a category containing the word *time*. Times entered are assumed to be a part of the business day unless *a* or *p* (for A.M. or P.M.) are added. Thus, if you enter 800 as the time, Appleworks will convert it to 8:00 A.M. Likewise, 3 will be converted to 3:00 P.M. 350 a would be converted to 3:50 A.M. If you enter a 24-hour time such as 1330, Appleworks converts it to a P.M. time (1:30 P.M).

Moving the Cursor in the File

The cursor is moved through the file using the arrow keys and OA-1 through OA-9.

Deleting Records

Get into the multiple-record layout (OA-Z). Put the cursor on the record you would like to delete. Press OA-D and AppleWorks will highlight the record the cursor is on. Use the arrow keys to highlight the records you wish to delete, then hit <R>.

In the single-record layout, move to the record you wish to delete and hit OA-D. You may delete several records by answering **yes** for each one you wish to delete. Hit <ESC> when you wish to stop deleting records.

Selecting Records

With AppleWorks, it is possible to set up selection rules so that only part of your file is printed. For example, you might want to print out all individuals in your file who are married. You may also select more than one category at a time (e.g., those in the blue reading group and who are male).

To change the selection rules, type OA-R (records). Select a *category* using the arrows or by typing a number and hit <R>. Select a *comparison* (e.g., if you want all those in the 90032 zip area, you would select **zip equals 90032.** If you would like to select based on a second category, select **and** or **or**, then choose a *category*, a *comparison* and a *value*. When you've entered the selection rules, press

<ESC>. For example, if you would like to select all records of people from the above file who are male and in the blue reading group, you would enter the following:

```
Selection: Sex equals male
  and      Reading Group equals blue
 <ESC>
```

To go back to All Records, type OA-R and select **yes.**

Finding Records

To use this option, usually, you want to be in the multiple-record layout. To find certain records, type OA-F, then type the sequence of characters you would like to find. Press <ESC> to leave this option.

Arranging Records

AppleWorks allows you to arrange records alphabetically, numerically, or chronologically. First, move the cursor to an entry in the category in which you wish to have your records arranged. Type OA-A (arrange), then select the way you want your records arranged.

Activity

1. Select all records of students who were born after 1/1/80 and who are Hispanic.
2. Return to **All Records**. Find all records containing a sequence of characters (e.g., "blue").
3. Sort your file alphabetically by last name. Sort it by birthdate (youngest to oldest).

Changing the Layout

The **layout** option allows you to move categories, change the width of columns, delete categories, etc. Type OA-L, then follow the directions that appear on the screen.

Changing a File's Structure

Type OA-N. If you wish to change the file's name, you may do so. If not, hit <R>. This will put you into the category names. Use the up and down arrows to move between the categories. When you are in the category you wish to change, use the editing features to make changes. Use OA-D to delete a category and OA-I to insert a category. When you are finished making changes, type <ESC>.

PRINTING A REPORT WITH THE DATA BASE

Table-Style Reports

A table-style report prints in rows and columns, similarly to a spreadsheet. First, type OA-P (print), next select **Create a new "tables" format.** Type the name of report (does not have to be the same as the name of the file). Now, the report format menu appears. You may move the cursor, (← & →), switch category positions (OA-> or OA-<), work within a category (OA-A), delete a category (OA-D), create a calculated category (you may have up to three in the data base program) (OA-G), right justify a category (OA-J), change a report name and/or title (OA-N), see printer options (OA-O), print the report (OA-P), and change record selection rules (OA-R). Note that when you change the format for a printout you are not altering the data base itself.

Activity

Use OA-Z to select the multiple-records display option. Type OA-P, then move the birthdate column to the end of the report. Next, delete the telephone number, marital status, and undergraduate college categories. Finally, print the report.

Label-Style Reports

Type OA-P, and select **Create a new "labels" format.** Type a name for the report and hit <R>. Now, the category names will appear. If you simply print the report at this point, only the category entries will appear (no spacing or formatting, one after the other). However, you may use the arrow keys to move the cursor to select categories to print or move, etc.

1. To move or arrange the categories on the page so that more than one appears on a line, so that they are indented, and so forth, use OA← to move them left, OA→ to move them right, OA-up arrow to move them up, and OA-down arrow for down.
2. If you have moved a category out of a line, type OA-D to delete the blank line. Use this same option to delete a category that you do not wish to print (that will not be deleted from the file). To insert a spacing line or to insert a category previously deleted, type OA-I.
3. You may left justify entries within a category by selecting OA-J after moving the cursor to the category you would like justified.
4. To print only those records that meet your specifications, type OA-R, then change the selection rules.
5. To print the category names along with the information in the categories, move the cursor to the appropriate category and type OA-V.
6. To change the name of the report, type OA-N.
7. To see what the report will look like, type OA-Z.

Activity

1. Prepare a label-style printout using your data base. Your report should have the following format:

```
Last Name              <First Name
Street Address       <City          <State
    <Zip
Telephone Number
Sex
Ethnicity
Reading Group
Math Group
```

   (Underlined categories indicate that the category name should be printed. Those with a "<" should be left-justified.)

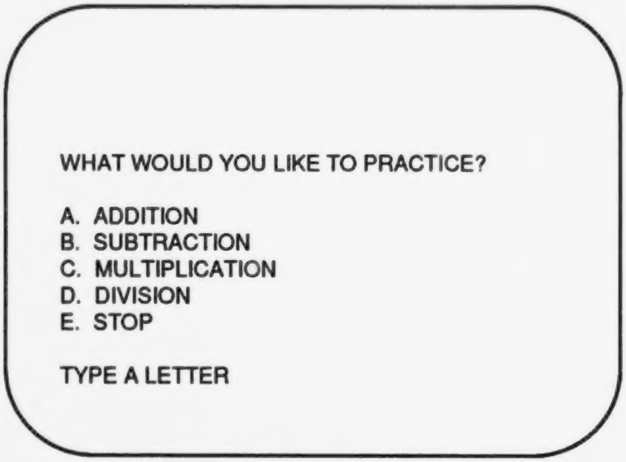
2. Print out the report.

APPENDIX

D The Math Practice Program

The program (Apple and IBM versions) listed in this appendix (and available on disc through the publisher: write to Gorsuch Scarisbrick Publishers, 8233 Via Paseo del Norte, Suite F400, Scottsdale, AZ 85258) is designed to give students practice in adding, subtracting, multiplying, and dividing whole numbers. When run, it first presents the user with a menu allowing her to choose the desired operation. Next, a menu allows the student to choose the kind of numbers to practice. Figures D.1 to D.5 show the different menu screens printed in this program.

FIGURE D.1

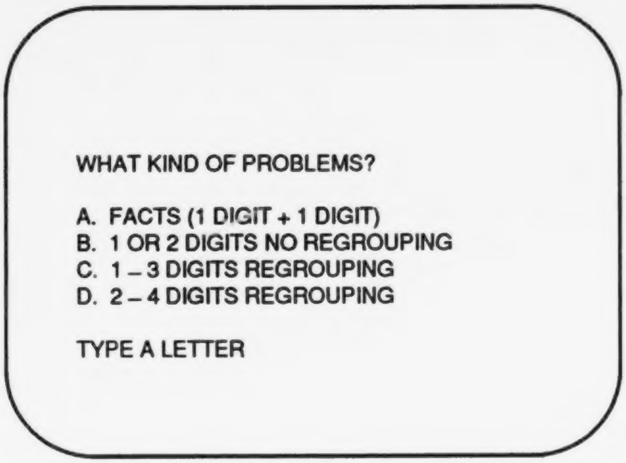


```
WHAT WOULD YOU LIKE TO PRACTICE?

A. ADDITION
B. SUBTRACTION
C. MULTIPLICATION
D. DIVISION
E. STOP

TYPE A LETTER
```

FIGURE D.2

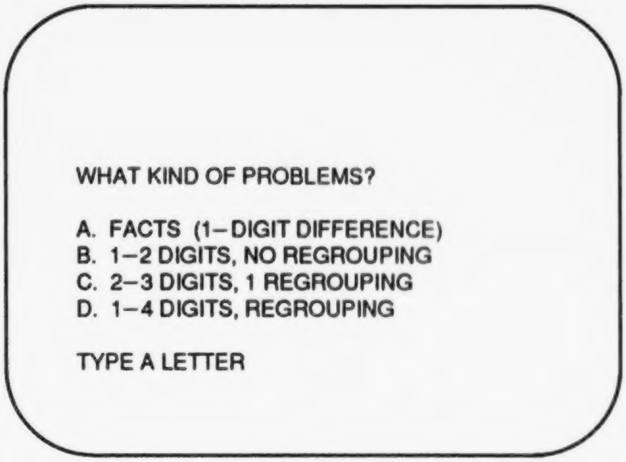


```
WHAT KIND OF PROBLEMS?

A.  FACTS (1 DIGIT + 1 DIGIT)
B.  1 OR 2 DIGITS NO REGROUPING
C.  1 – 3 DIGITS REGROUPING
D.  2 – 4 DIGITS REGROUPING

TYPE A LETTER
```

FIGURE D.3

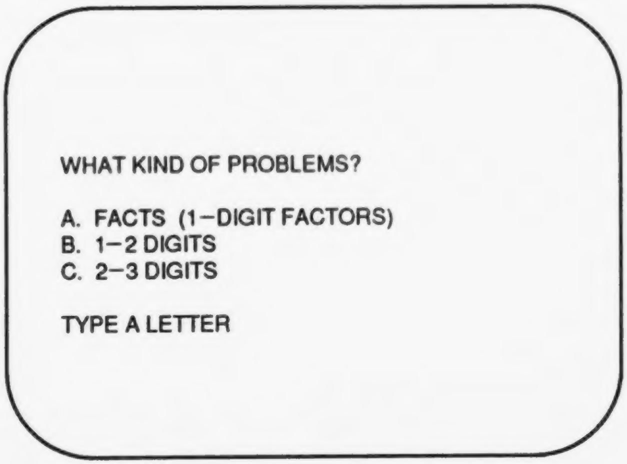


```
WHAT KIND OF PROBLEMS?

A.  FACTS  (1–DIGIT DIFFERENCE)
B.  1–2 DIGITS, NO REGROUPING
C.  2–3 DIGITS, 1 REGROUPING
D.  1–4 DIGITS, REGROUPING

TYPE A LETTER
```

FIGURE D.4

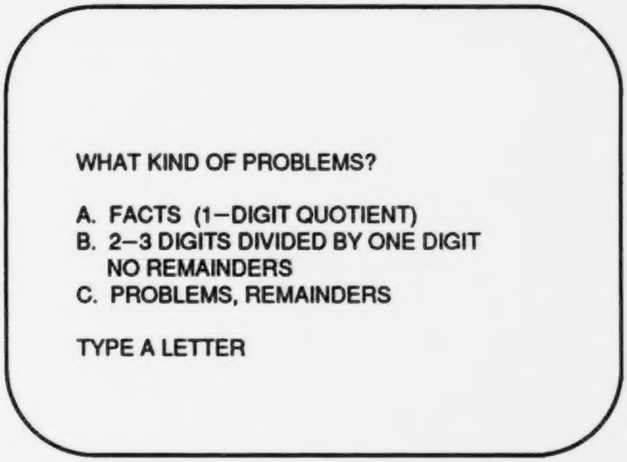


```
WHAT KIND OF PROBLEMS?

A.  FACTS  (1–DIGIT FACTORS)
B.  1–2 DIGITS
C.  2–3 DIGITS

TYPE A LETTER
```

FIGURE D.5



```
WHAT KIND OF PROBLEMS?

A. FACTS (1–DIGIT QUOTIENT)
B. 2–3 DIGITS DIVIDED BY ONE DIGIT
   NO REMAINDERS
C. PROBLEMS, REMAINDERS

TYPE A LETTER
```

```
10  REM ********************************************
20  REM *        THE MATH PRACTICE PROGRAM         *
30  REM *            BY JAMES H. WIEBE             *
35  REM *            APPLESOFT VERSION             *
40  REM ********************************************
50  REM
60  HOME
70  VTAB 8: HTAB 5
80  PRINT "WHAT WOULD YOU LIKE TO PRACTICE?"
85  PRINT
90  PRINT  TAB( 5)"A. ADDITION"
100  PRINT  TAB( 5)"B. SUBTRACTION"
110  PRINT  TAB( 5)"C. MULTIPLICATION"
120  PRINT  TAB( 5)"D. DIVISION"
125  PRINT  TAB( 5)"E. STOP"
130  PRINT
140  PRINT  TAB( 5)"TYPE A LETTER ";
150  GET A$
155  PRINT
160  IF ASC (A$) < 65 OR ASC (A$) > 69 THEN PRINT  TAB( 5)"PLEASE
     TYPE A, B, C, D, OR E": GOTO 150
170  A = ASC (A$) - 64
180  ON A GOSUB 2000,4000,6000,8000,10000
190  HOME : VTAB 8: HTAB 5
200  PRINT "DO YOU WANT TO TRY SOME MORE?"
205  PRINT  TAB( 5)"TYPE 'Y' OR 'N'"
210  HTAB 5
220  INPUT B$
230  IF B$ = "Y" THEN 60
240  END
1000  REM ***************************
1010  REM *     PROBLEM SET UP      *
1020  REM ***************************
1030  HOME
1040  VTAB 12: HTAB 15
1050  IF A = 1 THEN PRINT N1" + "N2" = ";
1052  IF A = 2 THEN PRINT N1" - "N2" - ";
1054  IF A = 3 THEN PRINT N1" X "N2" = ";
1060  INPUT AN
1070  RETURN
1200  REM *********************
1210  REM *    CHECK ANSWER   *
1220  REM *********************
```

```
1230  IF AN = (CA) THEN FLASH: VTAB 20: HTAB 15: PRINT "CORRECT":
      NORMAL: FOR ZZ = 1 TO 1500: NEXT ZZ: GOTO 1300
1240  VTAB 20: PRINT "                                        "
1250  VTAB 20: HTAB 13
1260  PRINT "WRONG, TRY AGAIN"
1262  FOR ZZ = 1 TO 1500: NEXT ZZ
1270  GOSUB 1000
1280  FOR ZZ = 1 TO 1000: NEXT ZZ
1290  IF AN = (CA) THEN VTAB 20: PRINT "
      ": VTAB 20: HTAB 18: PRINT "RIGHT": FOR ZZ = 1 TO 1500:
      NEXT ZZ: GOTO 1300
1294  VTAB 20: PRINT "                                          "
1296  VTAB 20: HTAB 10: PRINT "WRONG, THE ANSWER IS ";CA
1298  FOR ZZ = 1 TO 1500: NEXT ZZ
1300  RETURN
2000  REM  ***************************
2010  REM  *    ADDITION PROBLEMS    *
2020  REM  ***************************
2030  HOME
2040  VTAB 8: HTAB 5
2050  PRINT "WHAT KIND OF PROBLEMS?"
2055  PRINT
2060  PRINT  TAB( 5)"A. FACTS (1 DIGIT + 1 DIGIT)"
2070  PRINT  TAB( 5)"B. 1 OR 2 DIGITS NO REGROUPING"
2080  PRINT  TAB( 5)"C. 1 - 3 DIGITS REGROUPING"
2090  PRINT  TAB( 5)"D. 2 - 4 DIGITS REGROUPING"
2110  PRINT
2120  PRINT  TAB( 5)"TYPE A LETTER ";
2130  GET C$
2135  PRINT
2140  IF ASC (C$) < 65 OR ASC (C$) > 68 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, C, OR D": GOTO 2130
2142  PRINT
2143  PRINT  TAB( 5)"HOW MANY PROBLEMS";
2144  INPUT N
2150  C = ASC (C$) - 64
2160  IF C = 5 THEN 2200
2170  FOR X = 1 TO N
2180  ON C GOSUB 3000,3050,3100,3150
2190  GOSUB 1000
2200  CA = N1 + N2
2210  GOSUB 1200
2220  NEXT X
2250  RETURN
3000  REM  *** ADDITION FACTS ***
3010  N1 = INT (10 *  RND (1))
3020  N2 = INT (10 *  RND (1))
3030  RETURN
3050  REM  *** 1 - 2 DIGITS NO REG ***
3060  N1 = INT (60 *  RND (1))
3070  N3 = INT (2 *  RND (1)) + 1
3072  IF N3 = 1 THEN N2 =  INT (50 *  RND (1)): GOTO 3078
3074  N2 = INT (10 *  RND (1))
3078  IF N1 > 9 THEN N4 =  INT (N1 * .1) * 10:N4 = N1  - N4: GOTO
      3080
3079  N4 = N1
3080   IF N2 > 9 THEN N5 =  INT (N2 * .1) * 10:N5 = N2 - N5: GOTO
      3084
3082  N5 = N2
3084  IF N5 + N4 > 9 THEN 3060
3090  RETURN
3100  REM  *** 1 - 3 DIGITS ***
3102  N3 =  INT (5 *  RND (1)) + 1
3104  IF N3 < 3 THEN N1 =  INT (100 *  RND (1)): GOTO 3112
3110  N1 =  INT (490 *  RND (1)) + 9
3112  N3 =  INT (5 *  RND (1)) + 1
3114  IF N3 < 3 THEN N2 =  INT (100 *  RND (1)): GOTO 3130
```

```
3120  N2 =  INT (490 *  RND (1)) + 9
3130  RETURN
3150  REM  *** 2 - 4 DIGITS ***
3160  N3 =  INT (5 *  RND (1)) +1
3165  IF N3 < 3 THEN N1 =  INT (5900 *  RND (1)) + 100: GOTO 3175
3170  N1 =  INT (900 *  RND (1)) + 100
3175  N3 =  INT (5 *  RND (1)) + 1
3180  IF N3 < 4 THEN N2 =  INT (6000 *  RND (1)) + 100: GOTO 3190
3185  N2 =  INT (900 *  RND (1)) + 100
3190  RETURN
4000  REM  ***************************
4010  REM  *   SUBTRACTION PROBLEMS  *
4020  REM  ***************************
4030  HOME
4040  VTAB 8: HTAB 5
4050  PRINT  "WHAT KIND OF PROBLEMS?"
4060  PRINT
4070  PRINT  TAB( 5)"A. FACTS (1-DIGIT DIFFERENCE)"
4080  PRINT  TAB( 5)"B. 1-2 DIGITS, NO REGROUPING"
4090  PRINT  TAB( 5)"C. 2-3 DIGITS, 1 REGROUPING"
4100  PRINT  TAB( 5)"D. 1-4 DIGITS, REGROUPING"
4110  PRINT
4120  PRINT  TAB( 5)"TYPE A LETTER ";
4130  GET C$
4140  PRINT
4145  IF ASC (C$) < 65 OR ASC (C$) > 68 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, C, OR D": GOTO 4130
4147  PRINT
4148  PRINT  TAB( 5)"HOW MANY PROBLEMS";
4149  INPUT N
4150  C =  ASC (C$) - 64
4165  IF C = 5 THEN 4200
4170  FOR X = 1 TO N
4180  ON C GOSUB 5000,5050,5100,5150
4190  GOSUB 1000
4200  CA = N1 - N2
4210  GOSUB 1200
4220  NEXT X
4250  RETURN
5000  REM  *** SUBTRACTION FACTS ***
5010  N3 =  INT (10 *  RND (1))
5020  N2 =  INT (10 *  RND (1))
5030  N1 = N2 + N3
5040  RETURN
5050  REM  *** 1-2 DIGITS, NO REGROUP ***
5060  N3 =  INT (40 *  RND (1))
5063  N2 =  INT (30 *  RND (1))
5066  N1 = N2 + N3
5070  N4 = N1 -  INT (N1 * .1 + .000001) * 10
5073  N5 = N2 -  INT (N2 * .1) *10
5076  IF N4 < N5 THEN 5060
5080  RETURN
5100  REM  *** 2 - 3 DIGITS, 1 REGROUP ***
5110  N3 =  INT (300 *  RND (1))
5120  N2 =  INT (200 *  RND (1))
5123  N1 = N3 + N1
5126   N4 = N1 -  INT (N1 * .1 + .000001) * 10:N6 = N4 -  INT (N4
      * .01 + .0000001) * 100
5128  N5 = N2 -  INT (N2 * .1) * 10:N7 = N5 -  INT (N5 *  .01) *
      100
5130  IF N4 < N5 AND N6 < N7 THEN 5110
5140  RETURN
5150  REM  *** 2 - 4 DIGITS ***
5160  N3 =  INT (1990 *  RND (1)) + 10
5170  N2 =  INT (1990 *  RND (1)) + 10
5180  N1 = N2 + N3
5190  RETURN
```

```
6000  REM  *******************************
6010  REM  *   MULTIPLICATION PROBLEMS   *
6020  REM  *******************************
6030  HOME
6040  VTAB 8: HTAB 5
6050  PRINT "WHAT KIND OF PROBLEMS?"
6060  PRINT
6070  PRINT  TAB( 5)"A. FACTS (1-DIGIT FACTORS)"
6080  PRINT  TAB( 5)"B. 1-2 DIGITS"
6090  PRINT  TAB( 5)"C. 2-3 DIGITS"
6110  PRINT
6120  PRINT  TAB( 5)"TYPE A LETTER ";
6130  GET C$
6140  PRINT
6145  IF ASC (C$) < 65 OR ASC (C$) > 67 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, OR C": GOTO 6130
6147  PRINT
6148  PRINT  TAB( 5)"HOW MANY PROBLEMS ";
6149  INPUT N
6150  C = ASC (C$) - 64
6170  FOR X = 1 TO N
6180  ON C GOSUB 7000,7050,7100,7150
6190  GOSUB 1000
6200  CA = N1 * N2
6210  GOSUB 1200
6220  NEXT X
6250  RETURN
7000  REM  *** MULTIPLICATION FACTS ***
7010  N1 =  INT (10 *  RND (1))
7020  N2 =  INT (10 *  RND (1))
7030  RETURN
7050  REM  *** 1-2 DIGITS ***
7060  N1 =  INT (50 *  RND (1))
7070  N2 =  INT (50 *  RND (1))
7080  RETURN
7100  REM  *** 2-3 DIGITS ***
7110  N1 =  INT (400 *  RND (1)) + 10
7120  N2 =  INT (400 *  RND (1)) + 10
7130  RETURN
8000  REM  *****************************
8010  REM  *    DIVISION PROBLEMS      *
8020  REM  *****************************
8030  HOME
8040  VTAB 8: HTAB 5
8050  PRINT "WHAT KIND OF PROBLEMS?"
8060  PRINT
8070  PRINT  TAB( 5)"A. FACTS (1-DIGIT QUOTIENT)"
8080  PRINT  TAB( 5)"B. 2-3 DIGITS DIVIDED BY ONE DIGIT"
8085  PRINT  TAB( 5)"   NO REMAINDERS"
8090  PRINT  TAB( 5)"C. PROBLEMS, REMAINDERS"
8110  PRINT
8120  PRINT  TAB( 5)"TYPE A LETTER ";
8130  GET C$
8140  PRINT
8145  IF ASC (C$) < 65 OR ASC (C$) > 67 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, OR C": GOTO 8130
8147  PRINT
8148  PRINT  TAB( 5)"HOW MANY PROBLEMS ";
8149  INPUT N
8150  C = ASC (C$) - 64
8170  FOR X = 1 TO N
8180  ON C GOSUB 9000,9050,9100
8190  HOME
8195  N6 = 0
8200  VTAB 11: HTAB 17
8210  PRINT "_____"
8220  VTAB 12: HTAB 14
```

```
8230  PRINT N1: VTAB 12: HTAB 17
8240  PRINT ") ";
8242  IF N2 < 10 THEN PRINT "  ";N2
8244  IF N2 > 9 AND N2 < 100 THEN PRINT " ";N2
8246  IF N2 > 100 THEN PRINT N2
8249  IF C=3 THEN VTAB 14: PRINT "TYPE ANSWER, HIT RETURN": PRINT
      "THEN TYPE THE REMAINDER AND HIT RETURN"
8250  N6 = N6 + 1
8251  IF N3 < 10 THEN HTAB 19
8252  IF N3 > 9 THEN HTAB 18
8253  VTAB 10: INPUT AN: IF C = 3 THEN HTAB 23: VTAB 10: INPUT
      "REMAINDER ";RE
8260  CA = N3
8270   IF AN = (CA) AND RE = N5 AND N6 = 1 THEN FLASH : VTAB 20:
      HTAB 15: PRINT "CORRECT": NORMAL : FOR ZZ = 1 TO 1500: NEXT
      ZZ: GOTO 8380
8275   IF AN = (CA) AND RE = N5 AND N6 = 2 THEN VTAB 20: PRINT "
                                   ": VTAB 20: HTAB 18: PRINT "RIGHT":
      FOR ZZ = 1 TO 1500: NEXT ZZ: GOTO 8380
8280  VTAB 20: PRINT "                                     "
8290  VTAB 20: HTAB 13
8300  IF N6 = 1 THEN PRINT "WRONG, TRY AGAIN"
8303  IF N6 = 2 THEN HTAB 10: PRINT "WRONG, THE ANSWER IS  ";N3:
      IF C = 3 THEN HTAB 10: PRINT "REMAINDER ";RE
8304  IF N6 = 2 AND C = 3 THEN HTAB 10: PRINT "REMAINDER ";RE
8310  VTAB 22: PRINT "           TYPE ANY KEY TO CONTINUE";
8315  GET A$
8319  VTAB 20 :PRINT "                                        "
8320  VTAB 22
8322  PRINT "                                      "
8325  VTAB 10 :PRINT "                                        "
8327  IF N6 < 2 THEN 8250
8380  NEXT X
9000  REM  *** FACTS ***
9005  N5 = 0
9010  N3 =  INT (9 *  RND (1)) + 1
9020  N1 =  INT (9 *  RND (1)) + 1
9030  N2 = N1 * N3
9040  RETURN
9050  REM  *** 2 DIG/1 DIG, NO REMAINDER ***
9053  N3 =  INT (30 *  RND (1)) + 10
9055  N5 = 0
9056  N1 =  INT (9 *  RND (1)) + 1
9060  N2 = N1 * N3
9070  RETURN
9100  REM  *** DIVISION WITH REMAINDERS ***
9110  N3 =  INT (30 *  RND (1)) + 1
9112  N4 =  INT (20 *  RND (1)) + 1
9114  N1 =  INT (20 *  RND (1)) + 1
9115  IF N4 > = N1 THEN 9110
9116  N2 = N1 * N3 + N4
9118  N5 = N4
9120  RETURN
10000  END
```

```
10  REM  *******************************************
20  REM  *       THE MATH PRACTICE PROGRAM         *
30  REM  *            BY JAMES H. WIEBE            *
35  REM  *               IBM VERSION               *
36  REM  * NOTE: RUN WITH CAPS LOCK KEY DOWN       *
40  REM  *******************************************
50  REM
60  CLS
70  LOCATE 8,5
80  PRINT "WHAT WOULD YOU LIKE TO PRACTICE?"
```

```
85  PRINT
90  PRINT  TAB( 5)"A. ADDITION"
100  PRINT  TAB( 5)"B. SUBTRACTION"
110  PRINT  TAB( 5)"C. MULTIPLICATION"
120  PRINT  TAB( 5)"D. DIVISION"
125  PRINT  TAB( 5)"E. STOP"
130  PRINT
.40  PRINT  TAB( 5)"TYPE A LETTER ";
150  A$=INKEY$:IF A$="" THEN 150
155  PRINT
160   IF ASC (A$) < 65 OR ASC (A$) > 69 THEN PRINT TAB( 5)"PLEASE
    TYPE A, B, C, D, OR E": GOTO 150
170  A = ASC (A$) - 64
180  ON A GOSUB 2000,4000,6000,8000,10000
190  CLS:LOCATE 8,5
200  PRINT "DO YOU WANT TO TRY SOME MORE?"
205  PRINT  TAB( 5)"TYPE 'Y' OR 'N'"
210  LOCATE 10,5
220  INPUT B$
230  IF B$ = "Y" THEN 60
240  END
1000  REM  ****************************
1010  REM  *      PROBLEM SET UP       *
1020  REM  ****************************
1030  CLS
1040  LOCATE 12,30
1050  IF A = 1 THEN PRINT N1" + "N2" = ";
1052  IF A = 2 THEN PRINT N1" - "N2" = ";
1054  IF A = 3 THEN PRINT N1" X "N2" = ";
1060  INPUT AN
1070  RETURN
1200  REM  *********************
1210  REM  *    CHECK ANSWER    *
1220  REM  *********************
1230  IF AN = (CA) THEN LOCATE 20,33: PRINT "CORRECT": FOR ZZ = 1
      TO 1500: NEXT ZZ: GOTO 1300
1240  LOCATE 20,20 :PRINT"                                       "
1250  LOCATE 20,30
1260  PRINT "WRONG, TRY AGAIN"
1262  FOR ZZ = 1 TO 1500: NEXT ZZ
1270  GOSUB 1000
1280  FOR ZZ = 1 TO 1000: NEXT ZZ
1290   IF AN = (CA) THEN LOCATE 20,30 :PRINT "                  ":
     LOCATE 20,36: PRINT "RIGHT": FOR ZZ = 1 TO 1500: NEXT ZZ:
   GOTO 1300
1294  LOCATE 20,30: PRINT "                                     "
1296  LOCATE 20,30: PRINT "WRONG, THE ANSWER IS ";CA
1298  FOR ZZ = 1 TO 1500: NEXT ZZ
1300  RETURN
2000  REM  ****************************
2010  REM  *    ADDITION PROBLEMS     *
2020  REM  ****************************
2030  CLS
2040  LOCATE 8,5
2050  PRINT "WHAT KIND OF PROBLEMS?"
2055  PRINT
2060  PRINT  TAB( 5)"A. FACTS (1 DIGIT + 1 DIGIT)"
2070  PRINT  TAB( 5)"B. 1 OR 2 DIGITS NO REGROUPING"
2080  PRINT  TAB( 5)"C. 1 - 3 DIGITS REGROUPING"
2090  PRINT  TAB( 5)"D. 2 - 4 DIGITS REGROUPING"
2110  PRINT
2120  PRINT  TAB( 5)"TYPE A LETTER ";
2130  C$=INKEY$:IF C$="" THEN 2130
2135  PRINT
2140   IF ASC(C$)<65 OR ASC(C$)>68 THEN PRINT TAB( 5)"PLEASE TYPE
     A, B, C, OR D":GOTO 2130
2142  PRINT
```

```
2143  PRINT  TAB( 5)"HOW MANY PROBLEMS";
2144  INPUT N
2150  C = ASC (C$) - 64
2160  IF C = 5 THEN 2200
2170  FOR X = 1 TO N
2180  ON C GOSUB 3000,3050,3100,3150
2190  GOSUB 1000
2200  CA = N1 + N2
2210  GOSUB 1200
2220  NEXT X
2250  RETURN
3000  REM  *** ADDITION FACTS ***
3005 GOSUB 9500
3010  N1 = INT (10 *  RND (1))
3020  N2 = INT (10 *  RND (1))
3030  RETURN
3050  REM  *** 1 - 2 DIGITS NO REG ***
3055 GOSUB 9500
3060  N1 = INT (60 *  RND (1))
3070  N3 = INT (2 *  RND (1)) + 1
3072  IF N3 = 1 THEN N2 =  INT (50 *  RND (1)): GOTO 3078
3074  N2 = INT (10 *  RND (1))
3078  IF N1 > 9 THEN N4 =  INT (N1 * .1) * 10:N4 = N1  - N4: GOTO
3080
3079  N4 = N1
3080   IF N2 > 9 THEN N5 =  INT (N2 * .1) * 10:N5 = N2 - N5: GOTO
3084
3082  N5 = N2
3084  IF N5 + N4 > 9 THEN 3060
3090  RETURN
3100  REM  *** 1 - 3 DIGITS ***
3101 GOSUB 9500
3102  N3 =  INT (5 *  RND (1)) + 1
3104  IF N3 < 3 THEN N1 =  INT (100 *  RND (1)): GOTO 3112
3110  N1 =  INT (490 *  RND (1)) + 9
3112  N3 =  INT (5 *  RND (1)) + 1
3114  IF N3 < 3 THEN N2 =  INT (100 *  RND (1)): GOTO 3130
3120  N2 =  INT (490 *  RND (1)) + 9
3130  RETURN
3150  REM  *** 2 - 4 DIGITS ***
3155  GOSUB 9500
3160  N3 =  INT (5 *  RND (1)) +1
3165  IF N3 < 3 THEN N1 =  INT (5900 *  RND (1)) + 100: GOTO 3175
3170  N1 =  INT (900 *  RND (1)) + 100
3175  N3 =  INT (5 *  RND (1)) + 1
3180  IF N3 < 4 THEN N2 =  INT (6000 *  RND (1)) + 100: GOTO 3190
3185  N2 =  INT (900 *  RND (1)) + 100
3190  RETURN
4000  REM  **************************
4010  REM  *   SUBTRACTION PROBLEMS  *
4020  REM  **************************
4030  CLS
4040  LOCATE 8,10
4050  PRINT  "WHAT KIND OF PROBLEMS?"
4060  PRINT
4070  PRINT  TAB( 5)"A. FACTS (1-DIGIT DIFFERENCE)"
4080  PRINT  TAB( 5)"B. 1-2 DIGITS, NO REGROUPING"
4090  PRINT  TAB( 5)"C. 2-3 DIGITS, 1 REGROUPING"
4100  PRINT  TAB( 5)"D. 1-4 DIGITS, REGROUPING"
4110  PRINT
4120  PRINT  TAB( 5)"TYPE A LETTER ";
4130  C$=INKEY$:IF C$="" THEN 4130
4140  PRINT
4145  IF ASC (C$) < 65 OR ASC (C$) > 68 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, C, OR D": GOTO 4130
4147  PRINT
4148  PRINT  TAB( 5)"HOW MANY PROBLEMS";
```

```
4149  INPUT N
4150  C =  ASC (C$) - 64
4165  IF C = 5 THEN 4200
4170  FOR X = 1 TO N
4180  ON C GOSUB 5000,5050,5100,5150
4190  GOSUB 1000
4200  CA = N1 - N2
4210  GOSUB 1200
4220  NEXT X
4250  RETURN
5000  REM  *** SUBTRACTION FACTS ***
5005  GOSUB 9500
5010  N3 =  INT (10 *  RND (1))
5020  N2 =  INT (10 *  RND (1))
5030  N1 = N2 + N3
5040  RETURN
5050  REM  *** 1-2 DIGITS, NO REGROUP ***
5055  GOSUB 9500
5060  N3 =  INT (40 *  RND (1))
5063  N2 =  INT (30 *  RND (1))
5066  N1 = N2 + N3
5070  N4 = N1 -  INT (N1 * .1 + .000001) * 10
5073  N5 = N2 -  INT (N2 * .1) *10
5076  IF N4 < N5 THEN 5060
5080  RETURN
5100  REM  *** 2 - 3 DIGITS, 1 REGROUP ***
5105  GOSUB 9500
5110  N3 =  INT (300 *  RND (1))
5120  N2 =  INT (200 *  RND (1))
5123  N1 = N3 + N2
5126   N4 = N1 -  INT (N1 * .1 + .000001) * 10:N6 = N4 -  INT (N4
     * .01 + .0000001) * 100
5128  N5 = N2 -INT (N2 *.1) * 10:N7 = N5 - INT (N5 * .01) * 100
5130  IF N4 < N5 AND N6 < N7 THEN 5110
5140  RETURN
5150  REM  *** 2 - 4 DIGITS ***
5155  GOSUB 9500
5160  N3 =  INT (1990 *  RND (1)) + 10
5170  N2 =  INT (1990 *  RND (1)) + 10
5180  N1 = N2 + N3
5190  RETURN
6000  REM  *******************************
6010  REM  *   MULTIPLICATION PROBLEMS   *
6020  REM  *******************************
6030  CLS
6040  LOCATE 8,10
6050  PRINT "WHAT KIND OF PROBLEMS?"
6060  PRINT
6070  PRINT  TAB( 5)"A. FACTS (1-DIGIT FACTORS)"
6080  PRINT  TAB( 5)"B. 1-2 DIGITS"
6090  PRINT  TAB( 5)"C. 2-3 DIGITS"
6110  PRINT
6120  PRINT  TAB( 5)"TYPE A LETTER ";
6130  C$=INKEY$:IF C$="" THEN 6130
6140  PRINT
6145  IF ASC (C$) < 65 OR ASC (C$) > 67 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, OR CD": GOTO 6130
6147  PRINT
6148  PRINT  TAB( 5)"HOW MANY PROBLEMS ";
6149  INPUT N
6150  C = ASC (C$) - 64
6170  FOR X = 1 TO N
6180  ON C GOSUB 7000,7050,7100,7150
6190  GOSUB 1000
6200  CA = N1 * N2
6210  GOSUB 1200
6220  NEXT X
```

```
6250  RETURN
7000  REM  *** MULTIPLICATION FACTS ***
7005  GOSUB 9500
7010  N1 =  INT (10 *  RND (1))
7020  N2 =  INT (10 *  RND (1))
7030  RETURN
7050  REM  *** 1-2 DIGITS ***
7055  GOSUB 9500
7060  N1 =  INT (50 *  RND (1))
7070  N2 =  INT (50 *  RND (1))
7080  RETURN
7100  REM  *** 2-3 DIGITS ***
7105  GOSUB 9500
7110  N1 =  INT (400 *  RND (1)) + 10
7120  N2 =  INT (400 *  RND (1)) + 10
7130  RETURN
8000  REM  ******************************
8010  REM  *      DIVISION PROBLEMS      *
8020  REM  ******************************
8030  CLS
8040  LOCATE 8,10
8050  PRINT "WHAT KIND OF PROBLEMS?"
8060  PRINT
8070  PRINT  TAB( 5)"A. FACTS (1-DIGIT QUOTIENT)"
8080  PRINT  TAB( 5)"B. 2-3 DIGITS DIVIDED BY ONE DIGITS"
8085  PRINT  TAB( 5)"   NO REMAINDERS"
8090  PRINT  TAB( 5)"C. PROBLEMS, REMAINDERS"
8110  PRINT
8120  PRINT  TAB( 5)"TYPE A LETTER ";
8130  C$=INKEY$:IF C$=""THEN 8130
8140  PRINT
8145  IF ASC (C$) < 65 OR ASC (C$) > 67 THEN PRINT TAB( 5)"PLEASE
      TYPE A, B, OR C": GOTO 8130
8147  PRINT
8148  PRINT  TAB( 5)"HOW MANY PROBLEMS ";
8149  INPUT N
8150  C = ASC (C$) - 64
8170  FOR X = 1 TO N
8180  ON C GOSUB 9000,9050,9100
8182  IF N2<10 THEN PL=34
8183  IF N2>10 AND N2<100 AND N3>9  THEN PL=34
8184  IF N2>10 AND N2<100 AND N3<10 THEN PL=35
8185  IF N2>99 AND N3<10 THEN PL=36
8186  IF N2>99 AND N3>9 THEN PL=35
8190  CLS
8195  N6 = 0
8200  LOCATE 11,34
8210  PRINT "_____"
8220  LOCATE 12,30
8230  PRINT N1: LOCATE 12,34
8240  PRINT ")";N2
8250  LOCATE 14,1:N6 = N6 + 1
8251  IF C = 3 THEN PRINT "TYPE ANSWER, HIT RETURN":  PRINT "THEN
    TYPE THE  REMAINDER AND HIT RETURN": LOCATE 10,PL:INPUT AN:
   LOCATE 10,40: INPUT "R ";RE
8254  IF C = 1 OR C = 2 THEN LOCATE 10,PL: INPUT AN:RE = 0:N4 = 0
8260  CA = N3
8270  IF AN = (CA) AND RE = N5 AND N6 = 1 THEN LOCATE 20,30:
 PRINT "CORRECT": FOR ZZ = 1 TO 1500: NEXT ZZ: GOTO 8380
8275  IF AN = (CA) AND RE = N5 AND N6 = 2 THEN LOCATE 20,1: PRINT
      "                                   ": LOCATE 20,26: PRINT
 "RIGHT":GOTO 8310
8280  LOCATE 20,1: PRINT "                                     "
8290  LOCATE 20,26
8300   IF N6 = 1 THEN PRINT "WRONG, TRY AGAIN":LOCATE 10,32:PRINT
"          "
8303  IF N6 = 2 THEN LOCATE  20,26: PRINT  "WRONG, THE  ANSWER IS
";N3
```

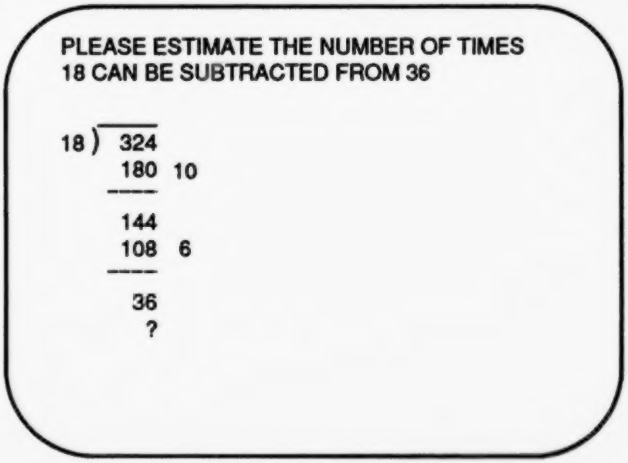
```
8304  IF N6  = 2  AND C  = 3 THEN LOCATE 21,26: PRINT "REMAINDER
";N4
8310  LOCATE 22,26 :PRINT "TYPE ANY KEY TO CONTINUE"
8315  A$=INKEY$:IF A$="" THEN 8315
8320  LOCATE 14,1
8324  FOR Z=1 TO 9:PRINT "
                         ":NEXT Z
8327  IF N6 < 2 THEN 8250
8380  NEXT X
9000  REM  *** FACTS ***
9005  GOSUB 9500
9010  N3 =  INT (9 *  RND (1)) + 1
9020  N1 =  INT (9 *  RND (1)) + 1
9030  N2 = N1 * N3
9040  RETURN
9050  REM  *** 2 DIG/1 DIG, NO REMAINDER ***
9051  GOSUB 9500
9053  N3 =  INT (30 *  RND (1)) + 10
9056  N1 =  INT (9 *  RND (1)) + 1
9060  N2 = N1 * N3
9070  RETURN
9100  REM  *** DIVISION WITH REMAINDERS ***
9101  GOSUB 9500
9110  N3 =  INT (30 *  RND (1))
9112  N4 =  INT (20 *  RND (1)) + 1
9114  N1 =  INT (20 *  RND (1))
9115  IF N4 > = N1 THEN 9110
9116  N2 = N1 * N3 + N4
9118  N5 = N4
9120  RETURN
9500  T=VAL(RIGHT$(TIME$,2))+60*VAL(MID$(TIME$,4,2))
9510  RANDOMIZE (T)
9520  RETURN
10000  END
```

APPENDIX

E Subtractive Algorithm Program

This appendix contains the Apple and IBM listings of the Subtractive Algorithm program that is on the disk available through the publisher (see Appendix D). The program places a problem on the screen, then asks the user to estimate the number of times the divisor can be subtracted from the dividend. The student types his estimate and the program multiplies this number by the divisor, subtracts the product from the dividend (or what is left after previous subtractions), and places the remaining amount on the screen. The program then asks the user to estimate the number of times the divisor can be subtracted from the remaining amount, and so on. If the student's estimate is too large, the program tells the student to try again. When the difference is zero, the program asks the student to add the number of times the divisor was subtracted from the dividend and write the answer (the quotient). Figure E.1 shows the screen of the IBM version of this program. Note that after the 6th subtraction, the program "wraps" to the top of the screen and continues the problem (see Figure E.2). Because of its 40-column screen (in BASIC) the Apple version of the program only wraps twice allowing up to 18 subtractions while the IBM version wraps four times allowing up to 30 subtractions.

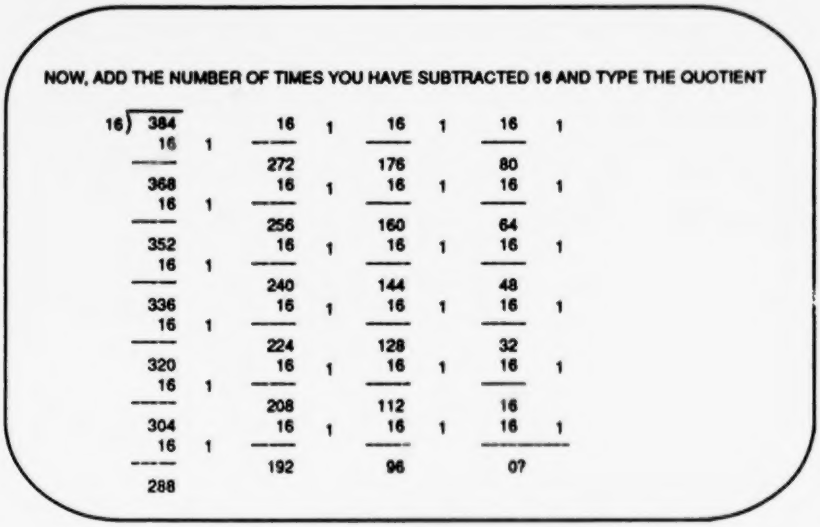
FIGURE E.1



```
PLEASE ESTIMATE THE NUMBER OF TIMES
18 CAN BE SUBTRACTED FROM 36

      ____
18 )  324
      180  10
     ----
      144
      108  6
     ----
       36
        ?
```

FIGURE E.2



```
NOW, ADD THE NUMBER OF TIMES YOU HAVE SUBTRACTED 16 AND TYPE THE QUOTIENT

16) 384          16    1      16    1      16    1
     16    1    ---          ---          ---
    ---          272          176           80
    368          16    1      16    1      16    1
     16    1    ---          ---          ---
    ---          256          160           64
    352          16    1      16    1      16    1
     16    1    ---          ---          ---
    ---          240          144           48
    336          16    1      16    1      16    1
     16    1    ---          ---          ---
    ---          224          128           32
    320          16    1      16    1      16    1
     16    1    ---          ---          ---
    ---          208          112           16
    304          16    1      16    1      16    1
     16    1    ---          ---          ---------
    ---          192           96           0?
    288
```

```
10  REM  ****************************************
20  REM  *     SUBTRACTIVE ALGORITHM PROGRAM    *
30  REM  *          BY JAMES H. WIEBE           *
35  REM  *          APPLESOFT VERSION           *
40  REM  ****************************************
50  HOME
60  VTAB 5
70  PRINT "THIS PROGRAM ALLOWS YOU TO PRACTICE"
80  PRINT "DIVISION ESTIMATIONS"
90  PRINT
100  PRINT "FOR EACH PROBLEM PLACED ON THE SCREEN,"
110  PRINT "ESTIMATE HOW MANY TIMES THE DIVISOR"
120  PRINT "(THE NUMBER YOU ARE DIVIDING BY)"
130  PRINT "MAY BE SUBTRACTED FROM THE DIVIDEND"
140  PRINT "(THE NUMBER YOU ARE DIVIDING INTO--"
150  PRINT "THE NUMBER UNDER THE DIVISION SIGN)"
160  PRINT
162  PRINT "HOW MANY PROBLEMS WOULD YOU LIKE";
164  INPUT N
170  FOR X = 1 TO N
180  GOSUB 1000
190  HOME
192  PRINT "PLEASE ESTIMATE THE NUMBER OF TIMES"
194  PRINT N1" CAN BE SUBTRACTED FROM ";N2
200  VTAB 4: HTAB 5
205  PRINT "______"
206  IF N2 > 99 THEN T = 7
207  IF N2 < 100 THEN T = 8
210  PRINT TAB( 2)N1; TAB( 5)")"; TAB( T)N2; TAB( 10)
220  VP = 3:HP = 10
225  N7 = N2
230  VP = VP + 3: IF VP > 22 AND HP = 10 THEN VP = 5:HP = 20
232  IF VP > 22 AND HP = 20 THEN VP = 5:HP = 30
235  VTAB VP: HTAB HP
240  INPUT " ";AN
243  IF AN < 10 THEN VTAB VP: HTAB HP: PRINT " "AN
244  IF AN * N1 > N7 THEN VTAB 3: PRINT "TOO BIG, TRY AGAIN": FOR ZZ =
     1 TO 1200: NEXT ZZ: VTAB 3: PRINT "                            ": VTAB
     VP: HTAB HP: PRINT "      ": VTAB VP: HTAB HP: GOTO 240
250  N5 = AN * N1
```

```
252  IF N5 > 99 THEN HTAB HP - 3
253  IF N5 > 9 AND N5 < 100 THEN HTAB HP - 2
254  IF N5 < 10 THEN HTAB HP - 1
256  VTAB VP: PRINT N5
260  N7 = N7 - N5
270  VTAB VP + 1: HTAB HP - 3: PRINT "---"
280  VTAB VP + 2
290  IF N7 > 99 THEN HTAB HP - 3
300  IF N7 > 9 AND N7 < 100 THEN HTAB HP - 2
310  IF N7 < 10 THEN HTAB HP - 1
320  PRINT N7
330  VTAB 2: HTAB 27: PRINT "    "
340  VTAB 2: HTAB 27: PRINT N7
350  IF N7 > 0 THEN 230
398  VTAB 1: PRINT "NOW, ADD THE NUMBER OF TIMES YOU HAVE"
400  PRINT "SUBTRACTED ";N1" AND TYPE THE QUOTIENT"
410  INPUT A9
420  IF A9 = N3 THEN VTAB 3: HTAB 12: FLASH : PRINT "CORRECT": NORMAL:
     FOR ZZ = 1 TO 1500: NEXT ZZ: GOTO 500
425  VTAB 3: HTAB 7
430  PRINT "SORRY, THE ANSWER IS "N3
440  FOR ZZ = 1 TO 2000: NEXT ZZ
500  NEXT X
990  NEXT X
995  END
1000  REM  *** RANDOMLY GENERATE NUMBERS ***
1010  N3 =  INT (30 *  RND (1)) + 1
1020  N1 =  INT (25 *  RND (1)) + 1
1030  N2 = N3 * N1
1035  IF N2 < 10 THEN 1010
1040  RETURN
```

```
100  REM **********************************************
110  REM *       SUBTRACTIVE ALGORITHM PROGRAM        *
120  REM *             BY JAMES H. WIEBE              *
130  REM *                IBM VERSION                 *
140  REM **********************************************
150  CLS
160  PRINT "THIS PROGRAM ALLOWS YOU TO PRACTICE DIVISION ESTIMATIONS."
170  PRINT
180  PRINT "FOR EACH PROBLEM PLACED ON THE SCREEN, ESTIMATE HOW MANY
     TIMES THE DIVISOR"
190  PRINT "(THE NUMBER YOU ARE DIVIDING BY) MAY BE SUBTRACTED FROM
     THE DIVIDEND"
200  PRINT "(THE NUMBER YOU ARE DIVIDING INTO--THE NUMBER UNDER THE
     DIVISION SIGN)"
210  PRINT
220  PRINT "HOW MANY PROBLEMS WOULD YOU LIKE ";
230  INPUT N
240  FOR X = 1 TO N
250  GOSUB 1000
260  CLS
270  PRINT "PLEASE ESTIMATE THE NUMBER OF TIMES "N1" CAN BE SUBTRACTED
     FROM ";N2
280  LOCATE 4,5
290  PRINT "_____"
300  IF N2 > 99 THEN T = 6
310  IF N2 < 100 THEN T = 7
320  PRINT N1; TAB( 5)")"; TAB( T)N2
330  VP = 3:HP = 9
340  N7 = N2
350  VP = VP + 3: IF VP > 22 AND HP = 9 THEN VP = 5:HP = 20
360  IF VP > 22 AND HP = 20 THEN VP = 5:HP = 30
370  IF VP> 22 AND HP=30 THEN VP=5:HP=40
380  IF VP>22 AND HP=40 THEN VP=5:HP=50
390  LOCATE VP,HP
```

```
400  INPUT " ";AN
410  IF AN < 10 THEN LOCATE VP,HP: PRINT "   ";AN
420  IF AN * N1 > N7 THEN LOCATE 3,1: PRINT "TOO BIG, TRY AGAIN": FOR
     ZZ = 1 TO 1200: NEXT ZZ: LOCATE 3,1: PRINT "
               ": LOCATE VP,HP: PRINT "      ": LOCATE VP,HP:GOTO 400
430  N5 = AN * N1
440  IF N5 > 99 THEN LP=HP-3
450  IF N5 > 9 AND N5 < 100 THEN LP=HP-2
460  IF N5 < 10 THEN LP=HP-1
470  LOCATE VP,LP: PRINT N5
480  N7 = N7 - N5
490  LOCATE VP+1,HP-2: PRINT "---"
500  IF N7 > 99 THEN LOCATE VP+2,HP-3
510  IF N7 > 9 AND N7 < 100 THEN LOCATE VP+2,HP - 2
520  IF N7 < 10 THEN LOCATE VP+2,HP - 1
530  PRINT N7
540  LOCATE 1,66: PRINT "    "
550  LOCATE 1,66: PRINT N7
560  IF N7 > 0 THEN 350
570  LOCATE 1,1
580   PRINT "
               "
590  LOCATE 1,1: PRINT "NOW, ADD THE NUMBER OF TIMES YOU HAVE
     SUBTRACTED ";N1;" AND TYPE THE QUOTIENT"
600  LOCATE VP+1,HP+2:PRINT "---"
610  LOCATE VP+2,HP+1
620  INPUT A9
630  IF A9<10 THEN LOCATE VP+2,HP+1:PRINT"   ":LOCATE VP+2,HP+3:
     PRINT A9
640  IF A9=N3 THEN LOCATE 2,35: PRINT "CORRECT":FOR ZZ=1 TO 1500:
     NEXT ZZ:GOTO 680
650  LOCATE 2,10
660  PRINT "SORRY, THE ANSWER IS "N3
670  FOR ZZ = 1 TO 2000: NEXT ZZ
680  NEXT X
690  END
1000  REM  *** RANDOMLY GENERATE NUMBERS ***
1010 T=VAL(RIGHT$(TIME$,2))+60*VAL(MID$(TIME$,4,2))
1020 RANDOMIZE (T)
1030 N3 =  INT (30 *  RND (1)) + 1
1040 N1 =  INT (25 *  RND (1)) + 1
1050 N2 = N3 * N1
1060  IF N2 < 10 THEN 1030
1070  RETURN
```

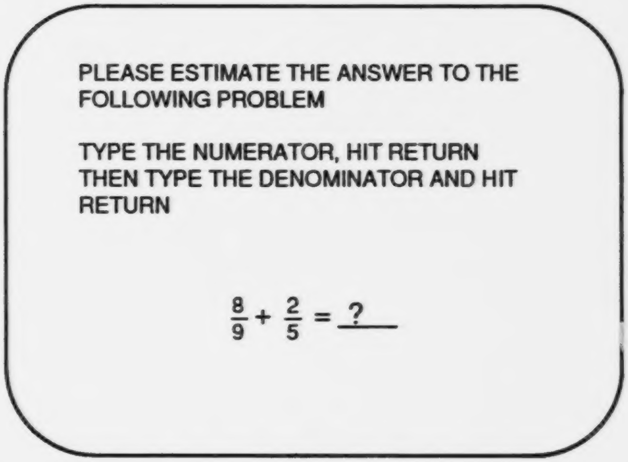
APPENDIX

F

Estimating Fractions Program

The program listed in this appendix and included on the disk available from the publisher (see Appendix D) is designed to give students practice in estimating answers to problems involving fractions. The program allows the user to select addition, subtraction, multiplication, or division problems. It places problems on the screen, one at a time (see Figure F.1), and asks the student to type the answer (the student types the numerator and pressses RETURN then types the denominator and presses RETURN). If the answer is within 50% less or 50% greater than the exact answer, it is considered to be correct. The student is given three tries to enter an acceptable answer. Exact answers to the questions are not printed, even after three incorrect inputs, in order to avoid giving students the idea that only one answer is correct since there are an unlimited number of possible correct responses. Of course, use of this program should follow appropriate instruction on estimating answers to operations on fractions.

FIGURE F.1



PLEASE ESTIMATE THE ANSWER TO THE FOLLOWING PROBLEM

TYPE THE NUMERATOR, HIT RETURN THEN TYPE THE DENOMINATOR AND HIT RETURN

$\frac{8}{9} + \frac{2}{5} = \underline{\ ?\ \ }$

Changing the range of acceptable answers in the program can be done by modifying line 620 of the program. For example, if you wanted to accept only answers within 33% of the exact answer, "AN / 2" in that line would be changed to "AN / 3." You can also easily change the fractions used in the problem to improper fractions or fractions involving larger denominations by changing the data statements. In the data statements, each pair of numbers is used to make a fraction (1 and 2 are used to make ½, 1 and 3 are used for ⅓, and so on).

```
10  REM  ****************************************
20  REM  *         ESTIMATING FRACTIONS         *
30  REM  *          BY JAMES H. WIEBE           *
35  REM  *          APPLESOFT VERSION           *
40  REM  ****************************************
45  DIM NN(40,2)
50  HOME
60  HTAB 5
70  PRINT "ESTIMATING FRACTIONS PROGRAM"
80  PRINT
90  VTAB 5: HTAB 5
100  PRINT "WHAT KIND OF PROBLEMS DO YOU WANT?"
110  PRINT
120  PRINT  TAB( 5)"A. ADDITION"
130  PRINT  TAB( 5)"B. SUBTRACTION"
140  PRINT  TAB( 5)"C. MULTIPLICATION"
150  PRINT  TAB( 5)"D. DIVISION"
160  PRINT
170  PRINT  TAB( 5)"TYPE A LETTER ";
180  GET A$
190  PRINT
200  IF  ASC (A$) < 65 OR  ASC (A$) > 69 THEN  PRINT  TAB( 5)"PLEASE
     TYPE A, B, C, OR D": GOTO 150
210  A =  ASC (A$) - 64
212  PRINT
215  HTAB 5
220  PRINT "HOW MANY PROBLEMS";
230  INPUT N
255  FOR Z = 1 TO 40
260  READ NN(Z,1),NN(Z,2)
265  NEXT Z
266  FOR X = 1 TO N
267  HOME
268  NT = 0
270  PRINT  TAB( 7)"PLEASE ESTIMATE THE ANSWER"
280  PRINT  TAB( 7)"TO THE FOLLOWING PROBLEM"
282  PRINT
284  PRINT  TAB( 7)"TYPE THE NUMERATOR, HIT RETURN"
286  PRINT  TAB( 7)"THEN TYPE THE DENOMINATOR"
288  PRINT  TAB( 7)"AND HIT RETURN"
290  VTAB 10: HTAB 13
300  R1 =  INT (26 *  RND (1)) +1
310  R2 =  INT (26 *  RND (1)) +1
320  IF A = 2 AND NN(R1,1) / NN(R1,2) <= NN(R2,1) / NN(R2,2)
     THEN 300
330  VTAB 10: HTAB 13
340  IF NN(R1,1) > 9 THEN  PRINT NN(R1,1): GOTO 360
350  PRINT " "NN(R1,1)
360  VTAB 10: HTAB 20
370  IF NN(R2,1) > 9 THEN  PRINT NN(R2,1): GOTO 390
380  PRINT " "NN(R2,1)
390  VTAB 11: HTAB 13
400  PRINT "--     -- = --"
410  VTAB 11: HTAB 17
420  IF A = 1 THEN  PRINT "+"
```

```
430  IF A = 2 THEN PRINT "-"
440  IF A = 3 THEN PRINT "X"
450  IF A = 4 THEN PRINT ":"
460  VTAB 12: HTAB 13
470  IF NN(R1,2) > 9 THEN PRINT NN(R1,2): GOTO 485
480  PRINT " "NN(R1,2)
485  VTAB 12: HTAB 20
490  IF NN(R2,2) > 9 THEN PRINT NN(R2,2): GOTO 510
500  PRINT " "NN(R2,2)
510  VTAB 10: HTAB 26
520  INPUT NU
530  IF NU < 10 THEN VTAB 10: HTAB 26: PRINT "   ": VTAB 10: HTAB
     28: PRINT NU: GOTO 535
532  VTAB 10: HTAB 26: PRINT " "
535  VTAB 12: HTAB 26
540  INPUT DE
550  IF DE < 10 THEN VTAB 12: HTAB 26: PRINT "   ": VTAB 12: HTAB
     28: PRINT DE: GOTO 570
560  VTAB 12: HTAB 26: PRINT " "
570  IF A = 1 THEN AN = NN(R1,1) / NN(R1,2) + NN(R2,1) / NN(R2,2)
580  IF A = 2 THEN AN = NN(R1,1) / NN(R1,2) - NN(R2,1) / NN(R2,2)
590  IF A = 3 THEN AN = NN(R1,1) / NN(R1,2) * (NN(R2,1) /
     NN(R2.2))
600  IF N = 4 THEN AN = NN(R1,1) / NN(R1,2) / (NN(R2,1) /
     NN(R2,2))
605  VTAB 20: HTAB 13
610  IF NU / DE > AN - AN / 2 AND NU / DE < AN + AN / 2 THEN
     FLASH : PRINT "GOOD ESTIMATE!": FOR ZZ = 1 TO 1500: NEXT ZZ:
     NORMAL : GOTO 800
615  NT=NT+1
620  IF NT < 3 THEN PRINT "WRONG, TRY  AGAIN": HTAB  26: VTAB 10:
     PRINT "    ":HTAB 26: VTAB 12: PRINT "    ":GOTO 510
630  PRINT "SORRY, THAT'S NOT RIGHT"
635  HTAB 13
640  PRINT "TRY ANOTHER PROBLEM"
650  FOR ZZ = 1 TO 1500: NEXT ZZ
800  NEXT X
990           DATA     1,2,1,3,2,3,1,4,3,4,1,5,2,5,3,5,4,5,1,6,5,
     6,1,8,3,8,5,8,7,8,1,9,2,9,4,9,5,9,7,9,8,9,
     1,10,3,10,5,10,7,10,9,10
995  DATA 3,2,5,2,4,3,5,3,5,4,7,4,6,5,7,5,8,5,7,
     9,8,11,8,10,9,11,10
```

```
10  REM  ***********************************************
20  REM  *              ESTIMATING FRACTIONS           *
25  REM  *                  IBM VERSION                *
30  REM  *               BY JAMES H. WIEBE             *
40  REM  ***********************************************
45  DIM NN(40,2)
50  CLS
70  PRINT TAB(26);"ESTIMATING FRACTIONS PROGRAM"
80  PRINT
90  LOCATE 5,10
100  PRINT "WHAT KIND OF PROBLEMS DO YOU WANT?"
110  PRINT
120  PRINT  TAB(10)"A. ADDITION"
130  PRINT  TAB(10)"B. SUBTRACTION"
140  PRINT  TAB(10)"C. MULTIPLICATION"
150  PRINT  TAB(10)"D. DIVISION"
160  PRINT
170  PRINT  TAB(10)"TYPE A LETTER ";
180  A$=INKEY$:IF A$="" THEN 180
190  PRINT
200  IF ASC (A$) < 65 OR (ASC (A$) > 69 AND ASC (A$) < 97) OR
     ASC (A$) > 100 THEN PRINT TAB( 5)"PLEASE TYPE A, B, C, OR
     D": GOTO 150
```

```
210  IF ASC(A$)<97 THEN A =  ASC (A$) - 64 ELSE A=ASC(A$)-96
212  PRINT
220  PRINT TAB(10);"HOW MANY PROBLEMS";
230  INPUT N
255  FOR Z = 1 TO 40
260  READ NN(Z,1),NN(Z,2)
265  NEXT Z
266  FOR X = 1 TO N
267  CLS
268  NT=0
270  PRINT  TAB( 7)"PLEASE ESTIMATE THE ANSWER TO THE FOLLOWING
     PROBLEM"
282  PRINT
284  PRINT  TAB( 7)"TYPE THE NUMERATOR, HIT RETURN THEN TYPE THE
     DENOMINATOR AND HIT RETURN"
290  LOCATE 10,26
295  T=VAL(RIGHT$(TIME$,2))+60*VAL(MID$(TIME$,4,2))
296  RANDOMIZE (T)
300  R1 =  INT (26 *  RND (1)) +1
310  R2 =  INT (26 *  RND (1)) +1
320  IF A = 2 AND NN(R1,1)/NN(R1,2)<=NN(R2,1)/NN(R2,2) THEN 300
330  LOCATE 10,33
340  IF NN(R1,1) > 9 THEN PRINT NN(R1,1): GOTO 360
350  PRINT " "NN(R1,1)
360  LOCATE 10,40
370  IF NN(R2,1) > 9 THEN PRINT NN(R2,1): GOTO 390
380  PRINT " "NN(R2,1)
390  LOCATE 11,34
400  PRINT "--     -- = --"
410  LOCATE 11,38
420  IF A = 1 THEN PRINT "+"
430  IF A = 2 THEN PRINT "-"
440  IF A = 3 THEN PRINT "X"
450  IF A = 4 THEN PRINT CHR$(246)
460  LOCATE 12,33
470  IF NN(R1,2) > 9 THEN PRINT NN(R1,2): GOTO 485
480  PRINT " "NN(R1,2)
485  LOCATE 12,40
490  IF NN(R2,2) > 9 THEN PRINT NN(R2,2): GOTO 510
500  PRINT " "NN(R2,2)
510  LOCATE 10,44
520  INPUT NU
530  IF NU < 10 THEN LOCATE 10,45: PRINT "   ": LOCATE 10,45:
     PRINT NU: GOTO 535
532  LOCATE 10,44: PRINT " "
535  LOCATE 12,44
540  INPUT DE
550  IF DE < 10 THEN LOCATE 12,45: PRINT "   ": LOCATE 12,45:
     PRINT DE: GOTO 570
560  LOCATE 12,44: PRINT " "
570  IF A = 1 THEN AN = NN(R1,1) / NN(R1,2) + NN(R2,1) / NN(R2,2)
580  IF A = 2 THEN AN = NN(R1,1) / NN(R1,2) - NN(R2,1) / NN(R2,2)
590  IF A = 3 THEN AN = NN(R1,1) / NN(R1,2) * (NN(R2,1) /
     NN(R2,2))
600  IF A = 4 THEN AN = NN(R1,1) / NN(R1,2) / (NN(R2,1) /
     NN(R2,2))
605  LOCATE 20,30
606  PRINT "                            "
607  LOCATE 20,30
610  IF NU / DE > AN - AN / 2 AND NU / DE < AN + AN / 2 THEN
     PRINT "GOOD ESTIMATE!": FOR ZZ = 1 TO 1500: NEXT ZZ: GOTO
     800
615  NT=NT+1
618  LOCATE 20,30
620  IF NT < 3 THEN PRINT "WRONG, TRY  AGAIN":LOCATE 10,44:PRINT
     "      ":LOCATE 12,44:PRINT "    ":GOTO 510
```

```
630  PRINT "SORRY, THAT'S NOT RIGHT"
635  LOCATE 21,30
640  PRINT "TRY ANOTHER PROBLEM"
650  FOR ZZ=1 TO 1500:NEXT ZZ
800  NEXT X
990  DATA 1,2,1,3,2,3,1,4,3,4,1,5,2,5,3,5,4,5,1,6,5,6,1,8,3,
     8,5,8,7,8,1,9,2,9,4,9,5,9,7,9,8,9,1,10,3,10,5,10,7,10,9,10
995  DATA 3,2,5,2,4,3,5,3,5,4,7,4,6,5,7,5,8,5,7,6,
     9,8,11,8,10,9,11,10
```

References

An Agenda for Action: Recommendations for School Mathematics of the 1980s. Reston, VA: National Council of Teachers of Mathematics, 1980.

Arithmetic Teacher (Special Calculator Issue), 34(6), February 1987.

Brownell, W. A. The place of meaning in the teaching of arithmetic. *Elementary School Journal* 47(5), January 1947: 256-65.

Bruner, J. S. On going beyond the information given. *Contemporary Approaches to Cognition.* Cambridge, MA: Harvard University Press, 1957.

Bruner, J. S. *The Process of Education.* Cambridge, MA: Harvard University Press, 1960.

Bruner, J. S. *Toward a Theory of Instruction.* Cambridge, MA: Belknap Press, 1971.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S. Jr., Lindquist, M. M. and Reys, R. E. *Results from the Second Mathematical Assessment of the National Assessment of Educational Progress.* Reston, VA: National Council of Teachers of Mathematics, 1981.

Computers in Mathematics Education (1984 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: National Council of Teachers of Mathematics, 1984.

Connolly, A. J., Nuchtman, W., and Prichett, E.M. *Key Math Arithmetic Diagnostic Test.* Circle Pines, MN: American Guidance Service, 1971.

Corbitt, M. K. The impact of computing technology on school mathematics: Report of an NCTM conference. *Arithmetic Teacher* 32(8), April 1985, 14–18.

Dienes, Z. P. and Golding, E. W. *Approach to Modern Mathematics.* New York: Herder and Herder, 1971.

Engelhardt, J. M. and Wiebe, J. H. An investigation of the underlying assumption of error analysis in Arithmetic. *Focus on Learning Problems in Mathematics* 3(1), January 1981: 13–24.

Forman, D. Search of the literature. *The Computing Teacher* 9(5), January 1982: 37–51.

Furth, H. G. On language and knowing in Piaget's developmental theory. *Human Development* 13(4), 1970: 241–257.

Kagen, J. Impulsive and reflective children: Significance of conceptual tempo. In *Learning and the Educational Process,* edited by J. Krumboltz. Chicago: Rand McNally, 1965.

Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve. Sacramento, CA: California State Department of Education, 1986.

Musser, G. L. Let's teach mental algorithms for addition and subtraction. *Arithmetic Teacher* 29(8), April 1982: 40–42.

(NAEP) National Assessment of Educational Progress. The Third National Assessment: Results, Trends and Issues. 13-MA-01. Denver, CO: Educational Commission of the United States, 1983.

Polya, G. *How to Solve It.* Garden City, NY: Doubleday Anchor Books, 1957.

Resnik, L. B. Task analysis in instructional design: Some cases from mathematics. In *Cognition and Instruction,* edited by D. Klahr. Hillsdale, NJ: Erlbaum, 1976, 51-80.

Reys, R. E., Bestgen, B. J., Rybolt, J. F. and Wyatt, J. W. Identification and characterization of computational estimation processes used by inschool pupils and out-of school adults. National Institute of Education (Grant No. 79-0088), November 1980.

Reys, R. E., Trafton, P. R., Reys, B., and Zawojewski, J. Developing computational estimation materials for the middle grades. National Science Foundation (Grant No. NSF 811360), March 1984a.

______. Developing computational estimation materials for the middle grades. Paper presented at the National Council of Supervisors of Mathematics, San Francisco, CA, April 1984b.

Riedesel, C. A., and Burns, P. C. Research on the teaching of elementary school mathematics. In *Second Handbook of Research on Teaching,* edited by R. M. Travers. Chicago: Rand McNally, 1973.

Roberts, D. M. The impact of electronic calculators on educational performance. *Review of Educational Research.* 50(1), Spring 1980: 71-98.

Romberg, T. A. and Carpenter, T. P. Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In *Handbook of Research on Teaching* (3rd Edition), edited by M. C. Wittrock. Chicago: Rand McNally, 1986.

Speer, W. R. "Do you see what I hear?" A look at individual learning styles. *Arithmetic Teacher* 27(3), November 1979: 22-26.

Steffe, L. P., vanGlassersfeld, E., Richards, J., and Cobbs, P. *Children's Counting Types: Philosophy, Theory and Applications.* New York: Praeger, 1983.

Suydam, M. N. Calculators in schools in the United States. In *International Calculator Review,* edited by M. N. Suydam. Columbus, OH: SMEAC Information Reference Center, 1980.

Suydam, M. N. Estimation and computation. *Arithmetic Teacher* 32(4), December 1984: 35.

Suydam, M. N. Update on research on problem solving: Implications for classroom teaching. *Arithmetic Teacher* 29(6), February 1982: 56-60.

Suydam, M. N. and Higgins, J. L. *Activity-based Learning in Elementary School Mathematics: Recommendaions from Research.* Columbus, OH: ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, 1977.

The Teaching of Arithmetic: Tenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

Trafton, P. The curriculum. In J. N. Payne (Ed.) *Mathematics Learning in Early Childhood* (Thirty-seventh Yearbook of the National Council of Teachers of Mathematics). edited by J. N. Payne. Reston, VA: National Council of Teachers of Mathematics, 1975.

Van Engen, H. An analysis of meaning in arithmetic. *Elementary School Journal.* 49(6 and 7), February/March 1949: 321-329, 395-400.

Weaver, J. F. and Suydam, M. N. *Meaningful Instruction in Mathematics Education.* Columbus, Ohio: ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, 1972.

Wheatly, C. L. Calculator use and problem-solving performance. *Journal for Research in Mathematics Education* 11(5), November 1980: 323-334.

Wiebe, J. H. Discovering fractions on a "fraction table." *Arithmetic Teacher* 33(4), December 1985: 49-51.

Wiebe, J. H. Needed: Good mathematics tutorial software for microcomputers. *School Science and Mathematics* 83(4), April 1983: 576-584.

Wiebe, J. H. Why do students have difficulties with percentages? A study of computational and error strategies. *Louisiana Educational Research Journal* 7(2), Summer 1982: 13-23.

INDEX

- Abacus, 118
- Absolute value, 340
- Addition:
  - decimals (*see* Decimals)
  - fractions (*see* Fractions)
  - integers (*see* Integers)
  - whole numbers, 141–156
    - associative property, 145–147; basic facts, 142–144; calculators, 141, 144, 145, 155; commutative (order) property, 145–146; computers, 143–144; estimation, 153–156; expanded notation, 150–151; identity property, 144–145; left-to-right, 152–153; multidigit, 147–156; partial sums algorithm, 151; verbal problems, 151
- *Agenda for Action, An*, 6, 31
- Algorithms, 23–24
  - decimals, 270–271
  - fractions, 1, 248–249, 252–254
  - front end, 40, 81–83
  - left to right, 81–83
  - pencil and paper, 1, 3, 7, 24, 26, 31, 42
  - whole numbers:
    - addition, 151; division, 3, 199, 207, 209; multiplication, 1, 24, 195; subtraction, 168–171
- Angles:
  - acute, 314
  - degrees, 310
  - exterior, 311, 314
  - interior, 311, 314
  - obtuse, 314
  - right, 314
  - supplementary, 314
- AppleWorks, 50–53, 56
  - data base, 52, 417–422
  - spreadsheet, 50–51, 56, 412–416
  - word processor, 52, 56
- Applications, 1, 5, 22, 68, 391
- Area (*see* Geometry)
- Array, 49, 179–180, 182, 226
- ASCII codes, 121
- Assessment, 357–363
  - error analysis, 360–362
  - interviews, 359–360
  - observation, 360
  - readiness, 358
  - tests, 358–360
- Associative property, 145–147, 187, 201, 333
- Attitudes, 31
- Attribute pieces, 106
- Average (*see* Mean)

- Back to Basics, 5
- Bar graph, 346–351
- Bases other than base ten, 5, 6, 47
  - base eight, 1, 47
  - base five, 121, 128
  - base sixteen, 1, 122
  - base two, 1, 47, 121, 125
- Base ten, 119–120
- Base ten blocks, 22, 125–126, 134, 147, 168, 228–229, 263–265
  - base two, 1, 47, 121, 125
- BASIC, 46, 48, 55, 89–90, 176, 187, 193, 211–212, 307, 316, 319, 354, 395–413
  - COLOR, 407
  - FOR-TO, NEXT, 405
  - GOTO, 402–403
  - graphics (coordinate), 49, 405–410
  - HCOLOR, 409–410
  - HGR, 409–410
  - HLIN, 408–409
  - HPLOT, 409–410
  - IF, 49, 403
  - INPUT, 404
  - LIST, 397
  - NEW, 397
  - PLOT, 406–407
  - PRINT, 144, 195, 204, 257, 339–340, 396–401
  - square root, 342
  - TAB, 400
  - True BASIC, 396
  - Turbo BASIC, 396
  - variables, 401–402
  - VLIN, 408–409
- Basic facts (*see also* Addition; Subtraction; Multiplication; Division), 40–41, 235, 378
  - memorization, 377–378
- Basic skills, 6
- Bean sticks, 22, 126
- Behavioral objectives, 364–366
- Binary operations, 145
- Bits, 120
- Boolean expressions, 49, 403
- Brownell, 16
- Bruner, Jerome, 12, 13, 19
- Bulletin boards (computer), 54
- Bundled sticks, 15, 22, 124–125, 147–148, 165–166, 175
- Byte, 121

- Calculators, 7–9, 19, 23–24, 26, 31–43, 85, 312, 324–325, 341, 354
  - with addition, 141, 144, 145, 151
  - algebraic entry, 42, 134
  - alphabet, 39
  - counting function, 32–33
  - and the curriculum, 3, 40–42, 239
  - with decimals, 235
  - developing meaning with, 32, 36–37
  - display, 42
  - with division, 34–35, 204–205, 208–209, 212
  - with estimation, 32, 37, 79–80
  - with fractions, 257
  - memory, 43, 212, 257
  - with multiplication, 186–187, 191, 195–196
  - with numbers and numeration, 110, 133–134, 136
  - percent key, 43, 282–283

  and practice, 38-39
  with problem solving, 37-38, 74
  with repeated estimation, 34-36
  reverse Polish, 42
  rounding, 205
  selection, 42-43
  solar powered, 42
  with subtraction, 163, 174
California Mathematics Framework, 6
Cardinal numbers, 105
Carrying (*see* Regrouping)
Cartesian products, 180, 181
Celsius, 301-302
Centimeter, 287, 296
Centimeter grid (*see* Grids)
Circle, 20, 309, 312
  circumference, 312
  radius, 312
Classification, 105-106
Classroom management, 50-54, 370-375
  managing materials, 373-375
  managing students, 370-373
COBOL, 395
Cognitive tempo (*see* Learning theories)
Colored chips, 22, 127-128, 138
Commutative property, 5, 19
  addition, 145-146
  division, 201
  multiplication, 186
  real numbers, 332
Composite numbers, 341
Computer assisted instruction, 56-60, 63-64
Computers, 1, 2, 8, 9, 23-24, 45-65, 79-80, 120-123, 207, 372, 395-443
  AppleWorks (*see* AppleWorks)
  BASIC (*see* BASIC)
  classroom management 50-54
  communications programs, 56
  compiled language, 395
  and the curriculum, 3, 45-49, 239, 324-325
  data bases (*see* Data base programs)
  disk operating system (DOS), 396
  fancy font programs, 52
  flowcharting, 46
  graphics, (*see* Graphics programs)
  hardware, 60
  interpreted language, 395
  Logo (*see* Logo)
  machine language, 3
  Pascal (*see* Pascal)
  problem solving, 46
  programming, 1, 46-49, 55, 74, 341
  spreadsheets (*see* Spreadsheets)
  telecommunications (*see* Telecommunications)
  word processors (*see* Word processing)
Concrete experiences (*see* Manipulatives)
Congruence, 322
Coordinate graphics (*see also* BASIC), 49
Correspondence (*see* One-to-one matching; One-to-many matching)
Counting, 104, 110
  counting back, 21
  counting on, 21, 145
  counting to, 21
Cubic units, 298-300
Cuisenaire rods, 159, 297-298
Curriculum (mathematics), 4
  history, 4-6
  impact of technology, 2, 3, 40-42, 45-49, 239, 324-325
Cylinder, 322

Data base programs, 52-53, 56, 63, 309-310, 322, 374
Data gathering, 345-352
Decimals:
  numerals:
    conversion to and from fractions, 41, 234-235; equivalent, 231-232, 273; importance in technological society, 46; manipulatives, 228-231; number line, 230-232; pictorial activities, 229-230; regrouping, 233; repeating, 234-235; rounding, 232-233; symbolic activities, 231-235; teaching with technology, 41
  operations, 261-271
    associative property, 262; commutative property, 262; distributive property, 262; estimation, 271-272; manipulatives, 263-267; number line, 263-268; symbolic activities, 267-271; zero property, 262
Degree, 310
Descriptive statistics, 352-355
  mean, 352
  median, 352-354
  mode, 352-353
  normal distribution, 352
  range, 354
  skewed distribution, 353
Diagnosis, 38, 58-59, 359, 362-363
Diagrams, 72-73
Dienes, Zolton, 13
Discovery learning, 11, 19, 20, 24-26
Disk operating system (DOS), 396
Distribution:
  normal, 352
  skewed, 353
Distributive property, 116, 188, 202, 333
Division:
  basic facts, 198-201
  decimals (*see* Decimals)
  fractions (*see* Fractions)
  whole numbers, 34, 35, 197-217
    algorithm (long division), 199, 207-209; associative property, 201; calculators, 34-35, 204-205, 208-209, 212; commutative (non), 201; computers, 204, 207; distributive, 202; estimation, 209-210; facts, 200-201; identity, 201; manipulatives, 198-200; measurement, 147, 198-199; multidigit, 202-212; number line, 200; partitive, 197-198; practice, 214-215; remainders, 204, 209; repeated subtraction, 202; subtractive algorithm, 205-207, 435-438; zero, 202
Dominoes, 111, 236
Drill and practice (*see* Practice)

Early mathematical experiences, 103-113
Egyptian number system (*see* Numeration systems)
Elastic, 23, 273-275, 278-280
Environment (mathematics in), 307, 309-310
Equation (*see* Open number sentences)
Equivalent fractions (*see* Fractions)
Equivalent ratios (*see* Proportion)
Error analysis, 360-362
Errors, 33, 360-362
Estimation, 24, 26-27, 75, 79-88, 153-156, 392
  addition (*see* Addition, whole numbers)
  averaging, 86-97
  and calculators, 3, 32, 40, 42
  compatible numbers, 83-86, 210
  compensation, 87
  fractions (operations), 256, 439-443
  front end digits, 40
  left-to-right, 81-83
  measurement, 298-299
  rounding, 83-86, 196, 210
Evaluation (*see* Assessment)
Expanded notation:
  and addition, 150-151
  and multiplication, 193-194
  numeration, 119, 132-133
  subtraction, 170
Exponent, 133

Face value, 119
Factorization (*see* Prime factorization)
Factors, 341
Factor trees, 341
Failure, nonmathematical causes, 362
Fancy font programs, 52
Field independence/dependence (*see* Learning theories)
Flip transformation, 327
Flow charting, 46, 72
Fraction bars, 223
Fraction pieces, 223
Fractions
  conceptual development, 219-231
    arrays, 226; converting to decimals, 41, 234-235; equivalence, 225-228;

    - improper, 223; manipulatives, 221-228; mixed numbers, 223; number lines, 227-228; numerals (names), 222; part-whole meaning, 220; subset meaning, 220, 224-225; and technology, 46
  - operations
    - addition and subtraction, like denominators, 240-243; addition and subtraction, unlike denominators, 249-254; calculators, 257; computers, 257, 439-443; division, 244-249; estimations, 86, 256, 439-443; least common denominator, 252-253; manipulatives, 240-248; mixed and improper, 254-256; multiplication and division, 242-249; and technology, 239.
- Frequency polygon (*see* Line graph)
- Frequency table, 346-347, 351
- Front end operations, 40, 173
- Future trends, 3, 6

- Games, 111-112, 137-139, 174-176, 212-214, 236-237, 257-258, 283-284, 378-389
  - teacher-made, 380-389
    - bingo, 380-382; board, 382-385; card, 385-387
- Geoboard, 23, 183, 314-316, 319
- Geometry;
  - angles (*see* Angles)
  - calculator, 312, 324-325
  - circle, 20, 309, 312
    - circumference, 312; radius, 312
  - computer (*see also* Logo), 307, 309-310
  - congruence, 322
  - environmental, 307, 309, 310
  - intuitive, 309
  - lines, 310
    - parallel, 318-319; of symmetry, 326
  - Logo, 49, 310-312, 314, 316, 319-321, 326
  - parallelogram, 321
  - planes, 316-317
  - point, 310
  - polygons, 321
  - ray, 310
  - rectangle, 320
  - regions, 317
  - rhombus, 321
  - segment, 311
  - shapes (two-dimensional), 309
  - solid geometry, 310, 321-322
    - surface area, 322-323; volumes, 322
    - square, 309, 320, 342
    - symmetry, 326-327
    - transformations:
      - rotations, 327; translations, 327
    - trapezoid, 321
    - triangles, 313-314, 320
- Gram, 287, 301
- Graphics programs, 52-53, 56, 63
- Graphing:
  - bar or histogram, 346-351
  - line or frequency polygon, 348-351
  - picture, 350-351
- Grids, centimeter, 23, 188, 189, 266, 277, 279, 297-298, 320
- Grouping (making sets), 105-106, 109
- Grouping students, 370-372

- Hindu-Arabic numbers (*see* Numeration systems)
- Histogram (*see* Bar graph)

- Identity element:
  - addition, 144-145
  - division, 201
  - multiplication, 186
  - real numbers, 333
- Individual differences, 23
- Individualized instruction, 50, 372-373
- Individual study, 377-378
- Integers, 331-340
  - addition and subtraction, 334-339
  - calculator, 339
  - computer, 339
  - multiplication and division, 339-340
  - negative, 333
  - number line, 333-341
- Invention, 21
- Irrational numbers, 332

- Karel the Robot, 46
- Kelvin scale, 286
- *Key Math Diagnostic Arithmetic Test*, 359, 376
- Kilogram, 287, 301

- Language and math, 12
- Learning theories, 20-21
  - discovery (*see* Discovery learning)
  - invention, 21
  - learning styles:
    - cognitive tempo, 20-21; field independence/dependence, 20
- Least common denominator, 252-253
- Least common multiple, 252-253
- Left-to-right computation, 81, 152-153, 171-172
- Length, 292-296
- Lesson planning, 363-368
- Light-emitting diode (LED), 42
- Line graph, 348-351
- Lines (*see* Geometry)
- Line segments, 311
- Liquid crystal display (LCD), 42
- Liter, 287, 300
- Location (terms of relative), 106-108
- Logo, 18, 19, 41, 46, 48-49, 55, 89-102, 110, 133, 176, 184, 193, 211-212, 307, 310-312, 314, 316-317, 319-321, 326, 354
  - BACK (BK), 91
  - CLEARSCREEN (CS), 91
  - editing, 92, 95
  - FORWARD (FD), 90
  - IF, 49, 55
  - immediate mode, 90-91
  - LEFT (LT), 91
  - MAKE, 55, 100
  - RIGHT (RT), 90
  - parentheses, 98-91
  - PENUP (PU), 91
  - PENDOWN (PD), 91
  - PRINT (PR), 55, 90, 97-99, 133, 144, 187, 195, 204, 257, 339-340
  - procedures, 90, 94-97, 100-101
  - procedures within procedures, 96-97
  - REPEAT, 55, 92-93, 312
  - square root, 342
  - Turtle Graphics, 55, 89-94, 96-97, 184
  - text, 97
  - variables, 99-101
- Lotus 1-2-3, 50, 54

- Machine language, 3, 123
- Manipulatives, 12-19, 21-23, 103
- Mass (weight), 301
- Matching (*see* One-to-one matching; One-to-many matching)
- Math corner, 27
- Math lab, 27
- Mean, 352
- Meaningful instruction, 16, 18
- Measurement, 285-305
  - area, 297-298
    - estimation, 298-299; grids, 298; intuitive, 297; tiling, 297
  - capacity, 300
  - early systems, 285
  - length, 294-296
  - mass, 301
  - money, 290-292
  - standard units, 285
  - teaching (general strategies), 288
  - temperature, 301-302
  - time, 292-294
    - digital, 293-295
  - U.S. customary system 286
  - volume, 298-300
    - formulas, 299-300; using unit cubes, 298-299; water displacement, 299-300
  - weight (mass), 301
- Media, 27, 352-354
- Median, 352-353
- Mental computation, 3, 7, 26, 42
- Mental images, 13-14
- Meter, 287, 296
- Metric Conversion Act, 286

Metric system of measurement, 286-288
  area, 297-298
  capacity, 300
  converting between units, 303, 305
  converting to U.S. Customary system, 304
  and the curriculum, 239
  linear, 296
  mass, 301
  prefixes, 287
  relationship between units, 287-288
  temperature, 301-302
  volume, 298-300
Microcomputers (*see* Computers)
Mira, 323-324, 326
Mode, 352-353
Models (*see* Manipulatives)
Modem, 53-54
Money, 40, 127, 132, 202, 290-292
Mousepaint, 53, 56
Multiple embodiments, 13
Multiplication
  basic facts, 18, 181-185
  decimals (*see* Decimals)
  fractions (*see* Fractions)
  integers (*see* Integers)
  of whole numbers, 116, 179-196
    algorithm, standard, 24, 195; array meaning, 179-180, 182-183; associative property, 187; basic facts, 181-185; using calculators, 184, 187, 195-196; Cartesian products, 180; commutative property, 186; using computers, 184, 187, 193, 196; distributive property, 188, 193-194; estimation, 196; identity property, 186; manipulatives, 181-183, 188-191; meanings, 179-180; multidigit, 188-196; number lines, 183; partial products, 23, 36, 191-192, 194-195; repeated addition, 24, 179, 182, 184, 191, 193; tables, 185, 201, 343; zero, 187

National Council of Supervisors of Mathematics, 7
National Council of Teachers of Mathematics, 7, 16
Networks, 318
New math, 5
Nonrational real numbers (*see* Real numbers, nonrational)
Number line, 23, 159-160, 183, 227-228, 230-232, 241-247, 251-252, 254, 263-268, 333-341
Numbers
  cardinal, 105
  composite, 341
  concepts, 105
  counting:
    counting back, 21; counting on, 21, 145; counting to, 21
  developing understanding:
    multidigit, 123-140; prenumber concepts, 105-108; singledigit, 108-110
  integers, 331-333
  large, 133
  place value (*see* Place value)
  prime, 341
  rational, 219, 332
  real, 332, 341, 342
  regrouping, 20, 120, 134
  rounding, 81, 135-136
  whole, 108-140
Number sentences (*see* Open number sentences)
Numeral (*see* Numbers)
Numeration, teaching (*see also* Numbers), 40, 123-140
Numeration blocks (*see* Base ten blocks)
Numeration systems, 104-105
  bases other than ten (*see* Bases)
  Egyptian, 117-118
  Hindu-Arabic, 105
  Roman, 115, 118-119
  properties of numerations systems, 119-120

Objectives (*see* Behavioral objectives)
Obtuse angle, 314
One-to-many matching, 108
One-to-one matching, 108
Open number sentences, 48, 74, 162, 199
Order of operations, 49, 211-212, 399
Ordinal numbers, 105
Overhead projector, 27, 136, 268, 274

Papert, Seymour, 46, 89
Parallel lines, 318-319
Parallelogram, 321
Parentheses 49, 98-99, 211-212, 399
Pascal, 46, 397
Patterns, 5, 73
Pencil and paper computations (*see* Algorithms)
Percent, 33, 86
  calculator, 282-283
  estimation, 283
  greater than 100%, 282
  manipulatives, 277-280
  numerals, 277
  symbolic, 280-283
Periods (in numbers), 120
Pi, 20, 25
Piaget, Jean, 12, 89
Picture graph, 350-351
Pictures (pictorial level), 110, 130-131, 224
PL/1, 395-396
Place value, 118-119, 124-131
  and calculators, 40, 133-134, 136
  and decimals, 229
  expanded notation, 132-133
  manipulatives, 116, 124-131
  regrouping and renaming, 134
  rounding, 135-136
  symbolic, 131-134
Place value charts, 22, 129-131, 137, 147-148, 162, 166-168, 176, 190, 203, 233
Plane, 316-317
Planning, 363-369
  long-range, 369
Point, 310
Polygon, 321
Practice (Drill), 7, 111-112, 137-139, 174-176, 212-214, 236-237, 257-259, 283-284, 377-392, 423-474
Prime factorization, 342
Prime numbers, 341
Probability:
  empirical, 354-355
  theoretical, 355
Problem solving, 6, 27, 58, 67-78
  calculator, 37-39, 74
  check answer, 75
  computer, 58
  diagram, 72-73
  money, 292
  nonroutine, 68, 76
  teaching strategies, 27, 69
    carry out plan, 72-75; check answer, 75-76; developing a plan, 70-72; drawing a graph, 72; reasonableness of answer, 75; restating the problem, 69; table, 72; understand the problem, 69; unnecessary information, 70
Program (*see* Computers)
Proportion, 272-276
Puzzle, 111-112, 139, 387-390
Pythagorean theorem, 315-316, 342-343

Range, 354
Ratio, 272
Rational numbers, 219, 332
Ray, 310
Readiness, 358
Real numbers, 332, 341-342
  nonrational real, 332
  proportions, 332-333
  square root, 342
Rectangle, 320
Regions, 317
Regrouping and renaming, 36, 120, 134
Research, 5, 14, 18, 19, 31, 37, 53, 69, 74, 80, 123, 367
Reverse Polish calculator entry (*see* Calculators)

Roman numerals (*see* Numeration systems)
Rotation, 327
Rounding, 81, 135-136

Segment, 311
Sentence, mathematical (*see* Open number sentences)
Sets, 5
Similar triangles, 324-325
Size, 108
Slide transformation, 327
Software evaluation, 59-62
Solar-powered calculator, 42
Spreadsheets, computer, 50-51, 56, 63, 176
Sputnik, 4
Square, 309, 320, 342
Statistics:
  descriptive, 346
  inferential, 346
  mean, 342
  median, 352-353
  mode, 352-353
  range, 354
Subtraction:
  decimals (*see* Decimals)
  fractions (*see* Fractions)
  integers (*see* Integers)
  with whole numbers, 156-178
    basic facts, 158-162; calculator, 163, 174; compensation, 164; computer, 166; equal additions, 169-171; expanded notation, 168; front end, 173; left-to-right, 171-172; missing addend, 156-159; number line, 159-160; properties, 163-164; relationship with addition, 160-161; set comparison meaning, 156-158; set separation meaning, 156-158; standard algorithm, 168-171; subset meaning, 156, 159; zero, 163
Subtractive algorithm (*see* Division)
Supplementary angles, 314
Surface area, 322-323
Symbols, 12, 105
Symmetry, 322-327

Talking calculator, 39
Technology (*see also* Calculators; Computers), 1, 3, 4, 8, 40
Television, 8
Testing, 8, 358-359
  diagnostic, 359
  standardized, 358-359
  teacher-made, 359-360
Textbooks, 5, 40, 46, 80, 369-370
Time, 292-294
  digital, 293-295
Transformations, 327
Translation, 327
Trapezoid, 321
Triangles, 313, 314, 320
Truncating, 81
Turtle graphics (*see* Logo)

Understanding, 5, 22
U.S. Customary units, 286, 304-305

Van Engen, 14
Variable, 48, 99-101, 401-402
Video disk, 8
Video tape, 8
VISICALC, 50
Volume, 298-300

Water displacement, 299-300, 322
Weight (mass), 301
Word processing, 56, 63

Zero, 19, 108-110, 120, 144, 187, 201

0-89787-520-6